Incompressible Navier-Stokes Equation as port-Hamiltonian systems: velocity formulation versus vorticity formulation

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Outline

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Aim

- Recast the '*vorticity–stream formulation*' of the 2D incompressible Navier-Stokes equations as a port-Hamiltonian system;
- Compare the '*vorticity–stream*' and '*velocity–pressure*' formulations;
- Apply the structure-preserving spatial-discretization Partitioned Finite Element Method (PFEM);
- Investigate numerical simulations with comparison against the lid-driven cavity benchmarks.

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Major differences with previous PFEM applications

- Modulated Stokes-Dirac structure;
- Differential constitutive relation (in space).

Conservation of mass:

$$\partial_t \rho(t, \boldsymbol{x}) + \mathrm{div}\left(\rho(t, \boldsymbol{x})\, \boldsymbol{v}(t, \boldsymbol{x})\right) = 0,$$

where ρ is the mass density and $\boldsymbol{v}$ is the particle velocity, at time t and point $\boldsymbol{x}$.

Isentropic Newtonian fluids

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Linear momentum evolution (for a Newtonian fluid):

$$\rho\, \left(\partial_t + \boldsymbol{v} \cdot \mathbf{grad}\right)\, \boldsymbol{v} = -\mathbf{grad}(P) + \mu\, \boldsymbol{\Delta} \boldsymbol{v} + (\lambda + \mu)\, \mathbf{grad}\left(\mathrm{div}\left(\boldsymbol{v}\right)\right),$$

where P is the static pressure, μ the dynamic viscosity and $\lambda + \frac{2}{3}\mu$ the bulk viscosity.

The latter is neglected under Stokes assumption: $\lambda = -\frac{2}{3}\mu$.

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Using $-\boldsymbol{\Delta} = \mathbf{curl}\,\mathbf{curl} - \mathbf{grad}\,\mathrm{div}$:

$$\rho\,\partial_t\boldsymbol{v} = -\rho\,(\boldsymbol{v}\cdot\mathbf{grad})\,\boldsymbol{v} - \mathbf{grad}(P) - \underbrace{\mu}_{\mu_c}\,\mathbf{curl}\left(\mathbf{curl}\left(\boldsymbol{v}\right)\right) + \underbrace{(\lambda+2\mu)}_{\mu_d = \frac{4}{3}\mu}\,\mathbf{grad}\left(\mathrm{div}\left(\boldsymbol{v}\right)\right).$$

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Mora *et al.* IFAC-WC (2020)

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With **density** ρ and **velocity** $\boldsymbol{v}$ as energy variables, the co-energy variables are:

$e_\rho := \delta_\rho \mathcal{H} = \frac{1}{2} \|\boldsymbol{v}\|^2 + \frac{P}{\rho} = h(\rho, \boldsymbol{v}),$ the **enthalpy**,

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Port-Hamiltonian system (when supplemented with boundary controls and collocated observations):

$$\begin{pmatrix} \partial_t \rho \\ \rho\, \partial_t \boldsymbol{v} \\ \boldsymbol{f}_c \\ f_d \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{div}\,(\rho\, \cdot) & 0 & 0 \\ -\rho\,\mathbf{grad} & \boldsymbol{\omega} \wedge \cdot & -\mathbf{curl} & \mathbf{grad} \\ 0 & \mathbf{curl} & 0 & 0 \\ 0 & \mathrm{div} & 0 & 0 \end{bmatrix} \begin{pmatrix} e_\rho \\ \boldsymbol{e}_{\boldsymbol{v}} \\ \boldsymbol{e}_c \\ e_d \end{pmatrix}, \quad \begin{array}{ll} e_\rho = h, & \boldsymbol{f}_c := \boldsymbol{\omega} = \mathbf{curl}\,(\boldsymbol{v}), \\ \boldsymbol{e}_{\boldsymbol{v}} = \boldsymbol{v}, & f_d := \mathrm{div}\,(\boldsymbol{v}), \\ & \\ \boldsymbol{e}_c = \mu_c \boldsymbol{f}_c, & e_d = \mu_d f_d. \end{array}$$

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Power balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = -\int_\Omega \mu_d\, f_d^2 - \int_\Omega \mu_c\, \boldsymbol{f}_c^2 + \int_{\partial\Omega} \left[\left(\mu_d\, \mathrm{div}\,(\boldsymbol{v}) - \rho\, e_\rho\right) \boldsymbol{e_v}\cdot\boldsymbol{n} + \mu_c\, \boldsymbol{\omega}\cdot(\boldsymbol{e_v}\wedge\boldsymbol{n}) \right].$$

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Incompressible and rotational fluids

Velocity–pressure formulation of *incompressible* flow is a particular case.
Incompressibility $\rightsquigarrow \rho \equiv \rho_0$ and $\operatorname{div}(\boldsymbol{v}) = 0$, thus:

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$$\mathcal{H}(\boldsymbol{v}) := \frac{1}{2} \int_\Omega \rho_0 \left\|\boldsymbol{v}\right\|^2,$$

satisfying the **power balance**:

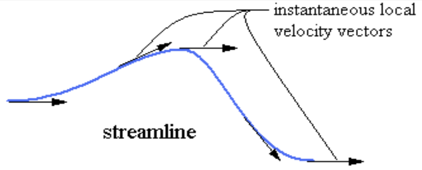
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = -\int_\Omega \mu_c\, \boldsymbol{f}_c^2 + \int_{\partial\Omega} \left[e_d\, \boldsymbol{e_v} \cdot \boldsymbol{n} + \mu_c \boldsymbol{\omega} \cdot (\boldsymbol{e_v} \wedge \boldsymbol{n}) \right].$$

The available boundary controls are the **normal** and **tangential** components of $\boldsymbol{v}$: $\boldsymbol{e_v} \cdot \boldsymbol{n}$ and $\boldsymbol{e_v} \wedge \boldsymbol{n}$.

Theorem.
In **2D**, if Ω is **simply connected**:

$$\forall\, \boldsymbol{v} \in \left(L^2(\Omega)\right)^2,\ \mathrm{div}\,(\boldsymbol{v}) = 0,\ \exists!\, \psi \in H^1(\Omega),$$

$$\text{such that} \quad \boldsymbol{v} = \mathbf{grad}^{\perp}(\psi) := \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix}.$$



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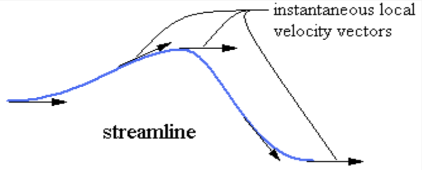

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Consequences on the $\mathbf{curl}$ operator, the vorticity $\boldsymbol{\omega}$ becomes a scalar field ω:

$$\boldsymbol{\omega} := \mathbf{curl} \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \partial_x v_2 - \partial_y v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \rightsquigarrow$$

$$\mathrm{curl}_{2D} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \partial_x v_2 - \partial_y v_1 = \omega,$$

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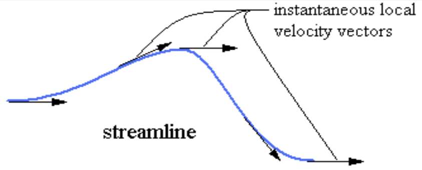

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The **integration by parts** then reads:

$$\int_\Omega \mathrm{curl}_{2D}(\boldsymbol{v})\ w = \int_\Omega \boldsymbol{v} \cdot \mathbf{grad}^\perp(w) + \int_{\partial\Omega} (\mathcal{R}\boldsymbol{v}) \cdot \boldsymbol{n}\,\gamma_0(w),$$

curl_{2D} and $\mathbf{grad}^\perp$ are formal adjoints!

where $\mathcal{R}$ is the rotation of angle $-\frac{\pi}{2}$ in the plane.

Incompressible and rotational fluids

Recall the **linear momentum** evolution (with $\rho \equiv \rho_0$ and $\operatorname{div}(\boldsymbol{v}) = 0$):

$$\rho_0 \left(\partial_t + \boldsymbol{v} \cdot \mathbf{grad}\right) \boldsymbol{v} = -\mathbf{grad}(P) + \mu \, \boldsymbol{\Delta} \boldsymbol{v}.$$

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$$\rho_0\,\partial_t\omega = \underbrace{\mathrm{div}\left(\omega\,\mathbf{grad}^{\perp}(\psi)\right)}_{J(\omega)\,\psi}\underbrace{-\mu_c\,\mathrm{curl}_{2D}\left(\mathbf{grad}^{\perp}(\omega)\right)}_{\mu_c\,\Delta\,\omega},\quad \text{using}\quad \begin{array}{l} \mathrm{curl}_{2D}\,\mathbf{grad} \equiv 0, \\ (\boldsymbol{v}\cdot\mathbf{grad}) = \mathbf{grad}\left(\frac{\|\boldsymbol{v}\|^2}{2}\right) + \boldsymbol{\omega}\wedge\boldsymbol{v}, \\ \mathrm{curl}_{2D}\left(\omega\wedge\mathbf{grad}^{\perp}(\psi)\right) = J(\omega)\,\psi. \end{array}$$

Remark that $J(\omega)$ is formally skew-symmetric (since $\mathbf{grad}\cdot\mathbf{grad}^{\perp} \equiv 0$).

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The **Hamiltonian** $\mathcal{H}(\omega) = \frac{1}{2} \int_\Omega \rho_0 \, \|\boldsymbol{v}\|^2$, **with** ω **as energy variable**, leads to the co-energy variable (Olver, 1993):

$$e_\omega := \delta_\omega^{\rho_0} \mathcal{H} = \psi.$$

Incompressible and rotational fluids

Thanks to $-\Delta = \operatorname{div} \mathbf{grad} = \operatorname{curl}_{2D} \mathbf{grad}^{\perp}$ and $\omega = -\Delta\psi$:

$$\rho_0 \, \partial_t \omega = J(\omega)\psi - \mu_c \left(\operatorname{curl}_{2D} \mathbf{grad}^{\perp} \right)^2 (\psi) \, .$$

Dissipative pHs under the '$J - R$' form, with $R = \mu_c \left(\operatorname{curl}_{2D} \mathbf{grad}^{\perp} \right)^2 = \mu_c \, \Delta^2$ the **bi-Laplacian**.

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allowing for the definition of a (modulated by the energy variable ω) Stokes-Dirac structure.

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Outline

A formal abstract class of distributed pHs

- The ***energy* variables** $\boldsymbol{\alpha}$;
- The **Hamiltonian** $\mathcal{H}(\boldsymbol{\alpha}(t))$;
- The ***co-energy* variables** $\boldsymbol{e}_{\boldsymbol{\alpha}} := \delta_{\boldsymbol{\alpha}}\mathcal{H}$ assumed to be given by $L\boldsymbol{\alpha}$;
- The ***structure* operator** J, *formally skew-symmetric*;
- The ***resistive/dissipative* operator** $R = GSG^\star$ and its related flow and effort $\boldsymbol{f}_R$ and $\boldsymbol{e}_R$;
- The ***explicit control* operator** $B_{\rm exp}$ (given by an operator $\gamma_{\rm exp}$, *e.g.* a boundary trace);
- The ***implicit control* operator** $B_{\rm imp}$ (given by an operator $\gamma_{\rm imp}$, *e.g.* a boundary trace);
- The **explicit input** $\boldsymbol{u}_{\rm exp}$ and the **explicit collocated output** $\boldsymbol{y}_{\rm exp}$;
- The **implicit input** $\boldsymbol{u}_{\rm imp}$ and the **implicit collocated output** $\boldsymbol{y}_{\rm imp}$.

Energy storage port →
Resistive port →
Boundary ports →
Constitutive relations →

$$\begin{pmatrix} \partial_t \boldsymbol{\alpha}(t) \\ \boldsymbol{f}_R(t) \\ \hdashline -\boldsymbol{y}_{\rm exp}(t) \\ \boldsymbol{u}_{\rm imp}(t) \end{pmatrix} = \begin{bmatrix} J & G & B_{\rm exp} & -B_{\rm imp} \\ -G^\star & 0 & & \\ \hdashline -B_{\rm exp}^\star & & 0 & 0 \\ B_{\rm imp}^\star & & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\alpha}}(t) \\ \boldsymbol{e}_R(t) \\ \hdashline \boldsymbol{u}_{\rm exp}(t) \\ -\boldsymbol{y}_{\rm imp}(t) \end{pmatrix},$$

$$\boldsymbol{e}_{\boldsymbol{\alpha}} = L\boldsymbol{\alpha}, \qquad \boldsymbol{e}_R = S\boldsymbol{f}_R,$$

satisfying an **abstract Green's formula** (formally):

$$\langle J\boldsymbol{e}_{\boldsymbol{\alpha}} + G\boldsymbol{e}_R, \phi \rangle = -\langle \boldsymbol{e}_{\boldsymbol{\alpha}}, J\phi\rangle + \langle \boldsymbol{e}_R, G^\star\phi\rangle + \left\langle \gamma_{\rm exp}\boldsymbol{e}_{\boldsymbol{\alpha}}, B_{\rm exp}^\star \phi \right\rangle + \left\langle \gamma_{\rm imp}\boldsymbol{e}_{\boldsymbol{\alpha}}, B_{\rm imp}^\star \phi \right\rangle.$$

Co-energy formulation and strategy

The **linear constitutive relations** can be taken into account inside the structure as follows:

Energy storage port →
Resistive port →
Boundary ports →

$$\begin{pmatrix} L^{-1}\partial_t \boldsymbol{e}_{\boldsymbol{\alpha}}(t) \\ S^{-1}\boldsymbol{e}_R(t) \\ \hdashline -\boldsymbol{y}_{\mathrm{exp}}(t) \\ \boldsymbol{u}_{\mathrm{imp}}(t) \end{pmatrix} = \left[\begin{array}{cc:cc} J & G & B_{\mathrm{exp}} & -B_{\mathrm{imp}} \\ -G^\star & 0 & & \\ \hdashline -B^\star_{\mathrm{exp}} & & 0 & 0 \\ B^\star_{\mathrm{imp}} & & 0 & 0 \end{array}\right] \begin{pmatrix} \boldsymbol{e}_{\boldsymbol{\alpha}}(t) \\ \boldsymbol{e}_R(t) \\ \hdashline \boldsymbol{u}_{\mathrm{exp}}(t) \\ -\boldsymbol{y}_{\mathrm{imp}}(t) \end{pmatrix},$$

Remark: taking the constitutive relations into account on the right-hand side (*energy formulation*) would raise numerical difficulties (such as matrix inversions).

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$$

Remark: taking the constitutive relations into account on the right-hand side (*energy formulation*) would raise numerical difficulties (such as matrix inversions).

The Partitioned Finite Element Method:

1. Write the variational formulation of the co-energy formulation;
2. Apply Green's formula on a partition of the system, in order to make the explicit control and the implicit observation appear;
3. Use conforming mixed finite element spaces of approximation.

Remark: *explicit* and *implicit* components in Green's formula are often exchangeable. It depends on the partition of the system, and has an influence on the finite element choices.

Outline

Application to the vorticity–stream function formulation

The co-energy formulation is arduous: the constitutive relation $\omega = -\Delta\psi = -\Delta e_\omega$ is **differential**!

$$\begin{pmatrix} -\rho_0 \, \Delta \partial_t e_\omega \\ \mu_c^{-1} \, e_c \end{pmatrix} = \begin{bmatrix} \mathrm{div}\left(\omega \, \mathbf{grad}^\perp \cdot\right) & -\mathrm{curl}_{2D} \, \mathbf{grad}^\perp \\ \mathrm{curl}_{2D} \, \mathbf{grad}^\perp & 0 \end{bmatrix} \begin{pmatrix} e_\omega \\ e_c \end{pmatrix}, \quad \begin{array}{l} \mathbf{grad}^\perp (e_\omega) \cdot \boldsymbol{n} = u_{\boldsymbol{n}}, \; y_{\boldsymbol{n}} = \gamma_0 (\omega e_\omega), \\ \mathbf{grad} (e_\omega) \cdot \boldsymbol{n} = u_\tau, \; y_\tau = -\gamma_0 (e_c), \\ u_{\mathrm{imp}} = \gamma_0 (e_\omega), \; y_{\mathrm{imp}} = \mathbf{grad} (e_c) \cdot \boldsymbol{n}. \end{array}$$

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The weak form:

$$\left\{ \begin{aligned} -\left\langle \rho_0\, \partial_t \Delta e_\omega, \phi \right\rangle &= \left\langle \mathrm{div}\left(\omega\, \left(\mathbf{grad}^\perp e_\omega\right)\right), \phi \right\rangle - \left\langle \mathrm{curl}_{2D}\left(\mathbf{grad}^\perp\left(e_c\right)\right), \phi \right\rangle, \\ \left\langle \mu_c^{-1}\, e_c, \xi \right\rangle &= \left\langle \mathrm{curl}_{2D}\left(\mathbf{grad}^\perp\left(e_\omega\right)\right), \xi \right\rangle, \\ \left\langle u_{\mathrm{imp}}, \theta_{\mathrm{imp}} \right\rangle &= \left\langle \gamma_0\left(e_\omega\right), \theta_{\mathrm{imp}} \right\rangle, \\ \left\langle y_\tau, \theta_\tau \right\rangle &= \left\langle -\gamma_0\left(e_c\right), \theta_\tau \right\rangle, \\ \left\langle y_{\boldsymbol{n}}, \theta_{\boldsymbol{n}} \right\rangle &= \left\langle \gamma_0\left(\omega e_\omega\right), \theta_{\boldsymbol{n}} \right\rangle. \end{aligned} \right.$$

Integration by parts for **each** second order differential operator, using $\mathcal{R}\,\mathbf{grad}^{\perp} = -\mathbf{grad}$:

$$\left\langle \mathrm{div}\left(\omega\,\mathbf{grad}^{\perp}\left(e_\omega\right)\right), \phi \right\rangle = -\left\langle \mu_c^{-1} e_c\,\mathbf{grad}^{\perp}\left(e_\omega\right), \mathbf{grad}\left(\phi\right) \right\rangle + \left\langle u_{\boldsymbol{n}}, \gamma_0\left(\mu_c^{-1} e_c\,\phi\right) \right\rangle,$$

$$-\left\langle \mathrm{curl}_{2D}\left(\mathbf{grad}^{\perp}\left(e_c\right)\right), \phi \right\rangle = -\left\langle \mathbf{grad}^{\perp}\left(e_c\right), \mathbf{grad}^{\perp}\left(\phi\right) \right\rangle + \left\langle y_{\mathrm{imp}}, \gamma_0\left(\phi\right) \right\rangle,$$

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The **new term** compare to usual linear continuous constitutive relations:

$$-\left\langle \rho_0\, \partial_t \Delta e_\omega, \phi \right\rangle = \left\langle \rho_0\, \mathbf{grad}\left(\partial_t e_\omega\right), \mathbf{grad}\left(\phi\right)\right\rangle - \left\langle \partial_t u_{\boldsymbol{\tau}}, \gamma_0\left(\rho_0\, \phi\right)\right\rangle.$$

Numerically, the *time derivative* of the tangent control is required!

Application to the vorticity–stream function formulation

Variables with the same index are **projected** on the same finite-dimensional space:

$$\boldsymbol{e}_\omega^{ap}(t,\boldsymbol{x}) = \sum_{j=1}^{N_\omega} e_\omega^j(t)\,\phi^j(\boldsymbol{x}) = \boldsymbol{\Phi}^\top(\boldsymbol{x}) \cdot \underline{e_\omega}(t),$$

$$\boldsymbol{e}_c^{ap}(t,\boldsymbol{x}) = \sum_{\ell=1}^{N_c} e_c^\ell(t)\,\xi^\ell(\boldsymbol{x}) = \Xi^\top(\boldsymbol{x}) \cdot \underline{e_c}(t),$$

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$$u_\tau^{ap}(t,\boldsymbol{x})u = \Theta_\tau^\top(\boldsymbol{x})\cdot \underline{u}_\tau(t), \qquad y_\tau^{ap}(t,\boldsymbol{x}) = \Theta_\tau^\top(\boldsymbol{x})\cdot \underline{y}_\tau(t),$$

$$\partial_t u_\tau^{ap}(t,\boldsymbol{x})u = \Theta_\tau^\top(\boldsymbol{x})\cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{u}_\tau(t), \qquad \left(y_\tau^{\mathrm{dt}}\right)^{ap}(t,\boldsymbol{x}) = \Theta_\tau^\top(\boldsymbol{x})\cdot \underline{y}_\tau^{\mathrm{dt}}(t),$$

$$u_{\boldsymbol{n}}^{ap}(t,\boldsymbol{x}) = \Theta_{\boldsymbol{n}}^\top(\boldsymbol{x})\cdot \underline{u}_{\boldsymbol{n}}(t), \qquad y_{\boldsymbol{n}}^{ap}(t,\boldsymbol{x}) = \Theta_{\boldsymbol{n}}^\top(\boldsymbol{x})\cdot \underline{y}_{\boldsymbol{n}}(t).$$

Application to the vorticity–stream function formulation

$$\mathrm{Diag}\begin{bmatrix} \mathbf{M}_{\rho_0} \\ \mathbf{M}_{\mu_c^{-1}} \\ \mathbf{M}_{\mathrm{imp}} \\ \mathbf{M}_{\boldsymbol{\tau}} \\ \mathbf{M}_{\boldsymbol{\tau}}^{\rho_0} \\ \mathbf{M}_{\boldsymbol{n}} \end{bmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_{\omega}(t) \\ \underline{e}_{c}(t) \\ \underline{u}_{\mathrm{imp}}(t) \\ -\underline{y}_{\boldsymbol{\tau}}(t) \\ -\underline{y}_{\boldsymbol{\tau}}^{\mathrm{dt}}(t) \\ -\underline{y}_{\boldsymbol{n}}(t) \end{pmatrix} = \begin{bmatrix} \mathbf{J}_{\boldsymbol{\omega}} & -\mathbf{G} & -\mathbf{B}_{\mathrm{imp}} & 0 & \mathbf{B}_{\boldsymbol{\tau},\mathrm{dt}} & \mathbf{B}_{\boldsymbol{n}} \\ \mathbf{G}^{\top} & 0 & 0 & -\mathbf{B}_{\boldsymbol{\tau}} & 0 & 0 \\ \mathbf{B}_{\mathrm{imp}}^{\top} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{B}_{\boldsymbol{\tau}}^{\top} & 0 & 0 & 0 & 0 \\ -\mathbf{B}_{\boldsymbol{\tau},\mathrm{dt}}^{\top} & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{B}_{\boldsymbol{n}}^{\top} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{e}_{\omega}(t) \\ \underline{e}_{c}(t) \\ -\underline{y}_{\mathrm{imp}}(t) \\ \underline{u}_{\boldsymbol{\tau}}(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{u}_{\boldsymbol{\tau}}(t) \\ \underline{u}_{\boldsymbol{n}}(t) \end{pmatrix},$$

$$
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\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_{\omega}(t) \\ \underline{e}_{c}(t) \\ \underline{u}_{\mathrm{imp}}(t) \\ -\underline{y}_{\tau}(t) \\ -\underline{y}_{\tau}^{\mathrm{d}t}(t) \\ -\underline{y}_{\boldsymbol{n}}(t) \end{pmatrix}
=
\begin{bmatrix}
\mathbf{J}_{\boldsymbol{\omega}} & -\mathbf{G} & -\mathbf{B}_{\mathrm{imp}} & 0 & \mathbf{B}_{\tau,\mathrm{d}t} & \mathbf{B}_{\boldsymbol{n}} \\
\mathbf{G}^{\top} & 0 & 0 & -\mathbf{B}_{\tau} & 0 & 0 \\
\mathbf{B}_{\mathrm{imp}}^{\top} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{B}_{\tau}^{\top} & 0 & 0 & 0 & 0 \\
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-\mathbf{B}_{\boldsymbol{n}}^{\top} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{pmatrix} \underline{e}_{\omega}(t) \\ \underline{e}_{c}(t) \\ -\underline{y}_{\mathrm{imp}}(t) \\ \underline{u}_{\tau}(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{u}_{\tau}(t) \\ \underline{u}_{\boldsymbol{n}}(t) \end{pmatrix},
$$

$$
\left(\mathbf{M}_{\rho_0}\right)_{i,j} = \int_{\Omega} \mathbf{grad}\left(\phi^{j}\right) \cdot \rho_0\, \mathbf{grad}\left(\phi^{i}\right), \qquad \left(\mathbf{M}_{\mu_c^{-1}}\right)_{k,\ell} = \int_{\Omega} \xi^{\ell}\, \mu_c^{-1}\, \xi^{k},
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Application to the vorticity–stream function formulation

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Application to the vorticity–stream function formulation

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$$(\mathbf{B}_{\tau})_{k,q} = \int_{\partial\Omega} \theta_{\tau}^q \, \gamma_0\left(\xi^k\right), \qquad (\mathbf{B}_{\tau,\mathrm{dt}})_{i,q} = \int_{\partial\Omega} \theta_{\tau}^q \, \gamma_0\left(\phi^i\right).$$

Outline

Simulations: The lid-driven cavity benchmarks

SCRIMP

Simulation and ContRol of Interactions in Multi-Physics

Actual developers: F. Monteghetti, A. Bendimerad-Hohl, G. Haine

Former developers: A. Brugnoli, A. Serhani, X. Vasseur

- **Python** programming language;
- **GMSH** - mesh generator;
- **FEniCS** - finite element library;
- **multiphenics** - mixed FE for FEniCS;
- **PETSc TS** - time integration of DAEs.

Benchmarks from http://www.zetacomp.com

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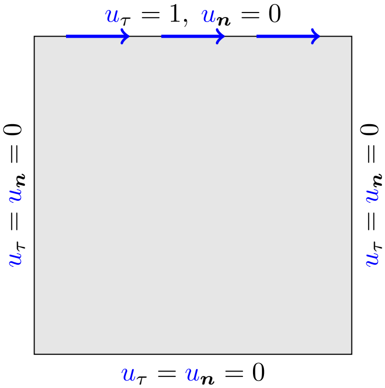
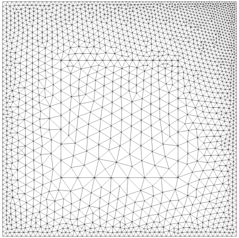
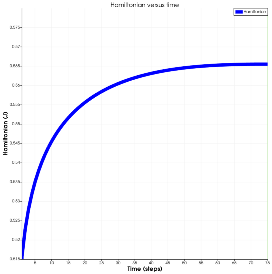


Simulations: The lid-driven cavity benchmarks

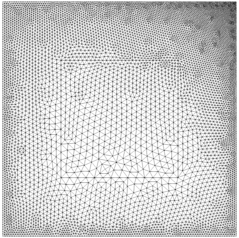
$\mu_c = 10^{-2}$ ($\sim$ Reynolds 10^2)

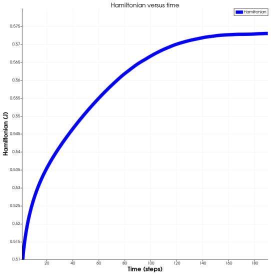




- $\mathbb{P}^2$ Lagrange elements for $e_\omega = \psi$;
- $\mathbb{P}^1$ Lagrange elements for $e_c = \omega$;
- $\mathbb{P}^1$ Lagrange elements at the boundary;
- 1 863 vertices, about 10 000 dofs;
- Backward Euler time scheme.

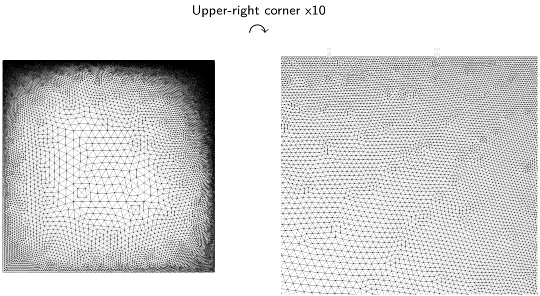
$$\mu_c = 2.5 * 10^{-3} \ (\sim \text{Reynolds } 4 * 10^2)$$





- $\mathbb{P}^2$ Lagrange elements for $e_\omega = \psi$;
- $\mathbb{P}^1$ Lagrange elements for $e_c = \omega$;
- $\mathbb{P}^1$ Lagrange elements at the boundary;
- 7 209 vertices, about 40 000 dofs;
- Backward Euler time scheme.

$$\mu_c = 10^{-3} \ (\sim \text{Reynolds } 10^3)$$



- $\mathbb{P}^3$ Lagrange elements for $e_\omega = \psi$;
- $\mathbb{P}^2$ Lagrange elements for $e_c = \omega$;
- $\mathbb{P}^1$ Lagrange elements at the boundary;
- 28 745 vertices, about 360 000 dofs;
- Backward Euler time scheme.

Outline

Conclusion and perspectives

Summary

- Successful port-Hamiltonian formulation of the vorticity–stream function formulation of a 2D isentropic and incompressible viscous fluid flow:
  - Modulated Stokes-Dirac structure (non-linearity);
  - Differential (in space) dissipative operator $R \sim \Delta^2$;
  - Differential (in space) constitutive relation $-\Delta$;
- Structure-preserving discretization;
- Efficient simulations compare against celebrated CFD benchmarks.

Further works

- Study the efficiency of the discretised vorticity–stream function formulation against the usual velocity–pressure formulation;
- Reduce the viscosity parameter ($\sim$ increase the Reynolds numbers);
- Investigate fluid stabilization thanks to the pHs formulation.

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The co-energy formulation:

$$\begin{pmatrix} \rho_0\,\partial_t \boldsymbol{e_v} \\ \mu_c^{-1}\,\boldsymbol{e}_c \\ u_{\mathrm{imp}} \end{pmatrix} = \begin{bmatrix} \mu_c^{-1}\,\boldsymbol{e}_c \wedge \cdot & -\mathbf{curl} & \mathbf{grad} \\ \mathbf{curl} & 0 & 0 \\ \mathrm{div} & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e_v} \\ \boldsymbol{e}_c \\ -y_{\mathrm{imp}} \end{pmatrix}, \quad \begin{matrix} \boldsymbol{e_v}\cdot\boldsymbol{n} = \gamma_{\boldsymbol{n}}\left(\boldsymbol{e_v}\right) = u_{\boldsymbol{n}}, & y_{\boldsymbol{n}} = \gamma_0\left(y_{\mathrm{imp}}\right), \\ \boldsymbol{e_v}\wedge\boldsymbol{n} = \gamma_\tau\left(\boldsymbol{e_v}\right) = \boldsymbol{u}_\tau, & \boldsymbol{y}_\tau = \boldsymbol{\gamma_0}\left(\boldsymbol{e}_c\right), \\ u_{\mathrm{imp}} = 0, & y_{\mathrm{imp}} = p + \frac{1}{2}\left\|\boldsymbol{v}\right\|^2. \end{matrix}$$

The weak form:

$$\begin{cases} \langle \rho_0\,\partial_t \boldsymbol{e_v}, \boldsymbol{\phi}\rangle = & \langle \mu_c^{-1}\,\boldsymbol{e}_c \wedge \boldsymbol{e_v}, \boldsymbol{\phi}\rangle \quad - \langle \mathbf{curl}\left(\boldsymbol{e}_c\right), \boldsymbol{\phi}\rangle \quad + \langle \mathbf{grad}\left(-y_{\mathrm{imp}}\right), \boldsymbol{\phi}\rangle, \\ \langle \mu_c^{-1}\,\boldsymbol{e}_c, \boldsymbol{\xi}\rangle = & \langle \mathbf{curl}\left(\boldsymbol{e_v}\right), \boldsymbol{\xi}\rangle, \\ \langle u_{\mathrm{imp}}, \theta_{\mathrm{imp}}\rangle = & \langle \mathrm{div}\left(\boldsymbol{e_v}\right), \theta_{\mathrm{imp}}\rangle, \\ \langle \boldsymbol{y}_\tau, \boldsymbol{\theta}_\tau\rangle = & \langle \boldsymbol{\gamma_0}\left(\boldsymbol{e}_c\right), \boldsymbol{\theta}_\tau\rangle, \\ \langle y_{\boldsymbol{n}}, \theta_{\boldsymbol{n}}\rangle = & \langle \gamma_0\left(y_{\mathrm{imp}}\right), \theta_{\boldsymbol{n}}\rangle. \end{cases}$$

Integration by parts of the second and third lines:

$$\begin{cases} \langle \rho_0\,\partial_t \boldsymbol{e_v}, \boldsymbol{\phi}\rangle = & \langle \mu_c^{-1}\,\boldsymbol{e}_c \wedge \boldsymbol{e_v}, \boldsymbol{\phi}\rangle \quad - \langle \mathbf{curl}\left(\boldsymbol{e}_c\right), \boldsymbol{\phi}\rangle \quad + \langle \mathbf{grad}\left(-y_{\mathrm{imp}}\right), \boldsymbol{\phi}\rangle, \\ \langle \mu_c^{-1}\,\boldsymbol{e}_c, \boldsymbol{\xi}\rangle = & \langle \boldsymbol{e_v}, \mathbf{curl}\left(\boldsymbol{\xi}\right)\rangle \quad + \langle \boldsymbol{u}_\tau, \boldsymbol{\gamma_0}\left(\boldsymbol{\xi}\right)\rangle, \\ \langle u_{\mathrm{imp}}, \theta_{\mathrm{imp}}\rangle = & -\langle \boldsymbol{e_v}, \mathbf{grad}\left(\theta_{\mathrm{imp}}\right)\rangle \quad + \langle u_{\boldsymbol{n}}, \gamma_0\left(\theta_{\mathrm{imp}}\right)\rangle, \\ \langle \boldsymbol{y}_\tau, \boldsymbol{\theta}_\tau\rangle = & \langle \boldsymbol{\gamma_0}\left(\boldsymbol{e}_c\right), \boldsymbol{\theta}_\tau\rangle, \\ \langle y_{\boldsymbol{n}}, \theta_{\boldsymbol{n}}\rangle = & \langle \gamma_0\left(y_{\mathrm{imp}}\right), \theta_{\boldsymbol{n}}\rangle. \end{cases}$$

Variables with the same index are **projected** on the same finite element bases:

$$e_{\boldsymbol{v}}^{ap}(t,\boldsymbol{x}) = \sum_j^{N_v} e_{\boldsymbol{v}}^j(t)\boldsymbol{\phi}^j(\boldsymbol{x}) = \boldsymbol{\Phi}^\top(\boldsymbol{x})\cdot \underline{e}_{\boldsymbol{v}}(t), \qquad e_c^{ap}(t,\boldsymbol{x}) = \boldsymbol{\Xi}^\top(\boldsymbol{x})\cdot \underline{e}_c(t),$$

$$u_{\rm imp}^{ap}(t,\boldsymbol{x}) = \Theta_{\rm imp}^\top(\boldsymbol{x})\cdot \underline{u}_{\rm imp}(t), \quad \boldsymbol{u}_{\boldsymbol{\tau}}^{ap}(t,\boldsymbol{x}) = \boldsymbol{\Theta}_{\boldsymbol{\tau}}^\top(\boldsymbol{x})\cdot \underline{u}_{\boldsymbol{\tau}}(t), \quad u_{\boldsymbol{n}}^{ap}(t,\boldsymbol{x}) = \Theta_{\boldsymbol{n}}^\top(\boldsymbol{x})\cdot \underline{u}_{\boldsymbol{n}}(t),$$

$$y_{\rm imp}^{ap}(t,\boldsymbol{x}) = \Theta_{\rm imp}^\top(\boldsymbol{x})\cdot \underline{y}_{\rm imp}(t), \quad \boldsymbol{y}_{\boldsymbol{\tau}}^{ap}(t,\boldsymbol{x}) = \boldsymbol{\Theta}_{\boldsymbol{\tau}}^\top(\boldsymbol{x})\cdot \underline{y}_{\boldsymbol{\tau}}(t), \quad y_{\boldsymbol{n}}^{ap}(t,\boldsymbol{x}) = \Theta_{\boldsymbol{n}}^\top(\boldsymbol{x})\cdot \underline{y}_{\boldsymbol{n}}(t),$$

$$\text{Diag}\begin{bmatrix} \mathbf{M}_{\rho_0} \\ \mathbf{M}_{\mu_c^{-1}} \\ \mathbf{M}_{\rm imp} \\ \mathbf{M}_{\boldsymbol{\tau}} \\ \mathbf{M}_{\boldsymbol{n}} \end{bmatrix} \begin{pmatrix} \frac{\rm d}{{\rm d}t}\underline{e}_{\boldsymbol{v}}(t) \\ \underline{e}_c(t) \\ \underline{u}_{\rm imp}(t) \\ -\underline{y}_{\boldsymbol{\tau}}(t) \\ -\underline{y}_{\boldsymbol{n}}(t) \end{pmatrix} = \begin{bmatrix} \mathbf{J}_{\boldsymbol{\omega}} & -\mathbf{C} & \mathbf{G} & 0 & 0 \\ \mathbf{C}^\top & 0 & 0 & \mathbf{B}_{\boldsymbol{\tau}} & 0 \\ -\mathbf{G}^\top & 0 & 0 & 0 & \mathbf{B}_{\boldsymbol{n}} \\ 0 & -\mathbf{B}_{\boldsymbol{\tau}}^\top & 0 & 0 & 0 \\ 0 & 0 & -\mathbf{B}_{\boldsymbol{n}}^\top & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{e}_{\boldsymbol{v}}(t) \\ \underline{e}_c(t) \\ -\underline{y}_{\rm imp}(t) \\ \underline{u}_{\boldsymbol{\tau}}(t) \\ \underline{u}_{\boldsymbol{n}}(t) \end{pmatrix},$$

$$(\mathbf{M}_{\rho_0})_{i,j} = \int_\Omega \boldsymbol{\phi}^j\cdot\rho_0\boldsymbol{\phi}^i, \quad \left(\mathbf{M}_{\mu_c^{-1}}\right)_{k,\ell} = \int_\Omega \boldsymbol{\xi}^\ell\cdot\mu_c^{-1}\boldsymbol{\xi}^k, \quad (\mathbf{M}_{\rm imp})_{m,n} = \int_\Omega \theta_{\rm imp}^n\,\theta_{\rm imp}^m,$$

$$(\mathbf{M}_{\boldsymbol{\tau}})_{p,q} = \int_{\partial\Omega} \boldsymbol{\theta}_{\boldsymbol{\tau}}^q\cdot\boldsymbol{\theta}_{\boldsymbol{\tau}}^p, \quad (\mathbf{M}_{\boldsymbol{n}})_{r,s} = \int_{\partial\Omega} \theta_{\boldsymbol{n}}^s\,\theta_{\boldsymbol{n}}^r,$$

$$(\mathbf{J}_{\boldsymbol{\omega}})_{i,j} = \int_\Omega \mu_c^{-1}\,\boldsymbol{e}_c^{ap}\cdot\left(\boldsymbol{\phi}^j\wedge\boldsymbol{\phi}^i\right), \quad (\mathbf{C})_{i,\ell} = \int_\Omega \mathbf{curl}\left(\boldsymbol{\xi}^\ell\right)\cdot\boldsymbol{\phi}^i, \quad (\mathbf{G})_{i,n} = \int_\Omega \mathbf{grad}\left(\theta_{\rm imp}^n\right)\cdot\boldsymbol{\phi}^i,$$

$$(\mathbf{B}_{\boldsymbol{\tau}})_{k,q} = \int_{\partial\Omega} \boldsymbol{\theta}_{\boldsymbol{\tau}}^q\cdot\boldsymbol{\gamma}_0\left(\boldsymbol{\xi}^k\right), \quad (\mathbf{B}_{\boldsymbol{n}})_{m,s} = \int_{\partial\Omega} \theta_{\boldsymbol{n}}^s\gamma_0\left(\theta_{\rm imp}^m\right).$$

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