Geometric Science of Information
21st–23rd July 2021, Paris

Structure-Preserving Discretization of a Coupled Heat-Wave System, as Interconnected Port-Hamiltonian Systems

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Simulate a coupled heat-wave system by using the *Partitioned Finite Element Method*, *and investigate the long-time behaviour of the approximated solution.*

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  $\rightarrow$ The power balance is *encoded* in a **Stokes-Dirac structure**.
- *Partitioned Finite Element Method (PFEM):*
  $\rightarrow$ It translates the Stokes-Dirac structure into a **Dirac structure**.
  $\rightarrow$ The **discrete Hamiltonian** satisfies the "discrete" power balance.

A Partitioned Finite Element Method for Power-Preserving Discretization of Open Systems of Conservation Laws
Cardoso-Ribeiro F.L., Matignon D. and Lefèvre L.
IMA Journal of Mathematical Control and Information, 38(2):493–533, (2020)

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$$\begin{bmatrix} Q^{-1} & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \partial_t \overrightarrow{e}_{\overrightarrow{\alpha}}(t) \\ \overrightarrow{e}_R(t) \\ -\boldsymbol{y}(t) \end{bmatrix} = \begin{bmatrix} J & -G & B \\ G^* & 0 & 0 \\ -B^* & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{e}_{\overrightarrow{\alpha}}(t) \\ \overrightarrow{e}_R(t) \\ \boldsymbol{u}(t) \end{bmatrix}.$$

Lossy Power Balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\overrightarrow{\alpha}(t)) = -\left\langle RQ\overrightarrow{\alpha}(t), Q\overrightarrow{\alpha}(t)\right\rangle_J + \langle \boldsymbol{u}(t), \boldsymbol{y}(t)\rangle_B \leq \langle \boldsymbol{u}(t), \boldsymbol{y}(t)\rangle_B .$$

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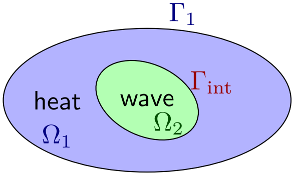
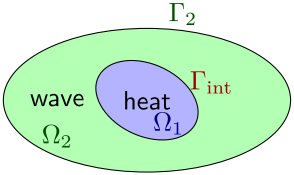
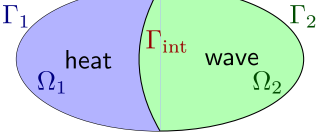
System and configurations







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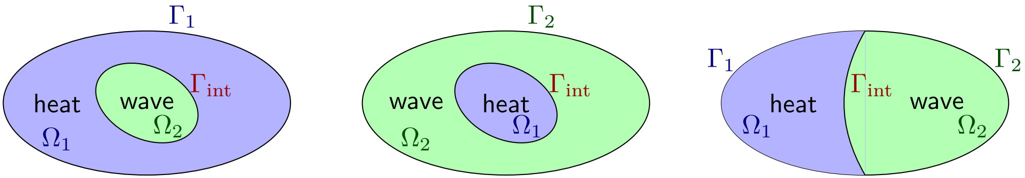

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$$\begin{cases} \partial_t T(t, \vec{x}) - \Delta T(t, \vec{x}) & = & 0, & \vec{x} \in \Omega_1, \\ T(t, \vec{x}) & = & 0, & \vec{x} \in \Gamma_1, \end{cases}$$

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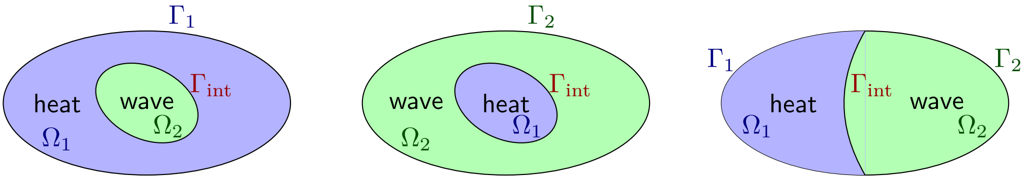

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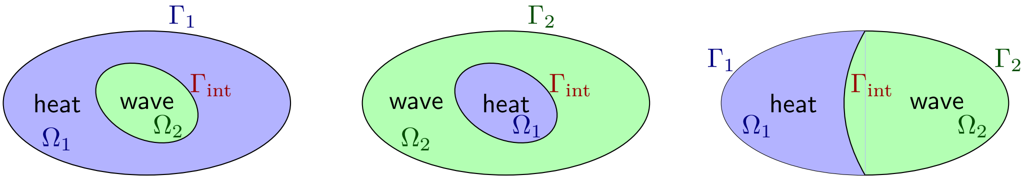

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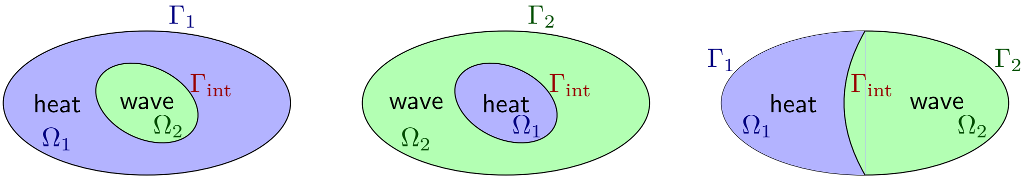

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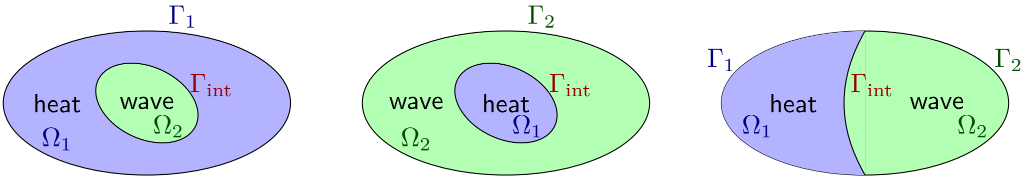

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Geometric Control Condition (GCC):
All characteristics of the wave equation must encountered Ω_1 in finite time.

Known decays

Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction

Zhang X. and Zuazua E.

Archive for Rational Mechanics and Analysis, 184(1):49–120, (2007)

Well-posedness (Zhang and Zuazua 2007)

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- Every solution is strongly stable (to zero if Γ_2 is non-empty, to a constant solution otherwise).
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Remark

In our numerical simulations, the initial data are such that the constant solution is the null solution.

Hamiltonian: $\mathcal{H}_T(T) := \frac{1}{2}\int_{\Omega_1} \left|T(t, \overrightarrow{\boldsymbol{x}})\right|^2 \mathrm{d}\overrightarrow{\boldsymbol{x}}$.

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_T = -\int_{\Omega_1} \left\|J_Q\right\|^2 + \left\langle \boldsymbol{u}_1, \boldsymbol{y}_1 \right\rangle_{H^{-\frac{1}{2}}(\Gamma_{\mathrm{int}}), H^{\frac{1}{2}}(\Gamma_{\mathrm{int}})}.$$

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The **total Hamiltonian** of the coupled Heat-Wave system is given by:

$$\mathcal{H}(T, \partial_t w, \nabla w) := \underbrace{\frac{1}{2} \int_{\Omega_1} \left|T(t, \overrightarrow{\boldsymbol{x}})\right|^2 \mathrm{d}\overrightarrow{\boldsymbol{x}}}_{\mathcal{H}_T} + \underbrace{\frac{1}{2} \int_{\Omega_2} \left|\partial_t w(t, \overrightarrow{\boldsymbol{x}})\right|^2 + \left\|\nabla w(t, \overrightarrow{\boldsymbol{x}})\right\|^2 \mathrm{d}\overrightarrow{\boldsymbol{x}}}_{\mathcal{H}_w}.$$

Coupling: transmission condition

Gyrator interconnection on Γ_{int}:

$$\boldsymbol{u}_1(t, \overrightarrow{\boldsymbol{x}}) = -\boldsymbol{y}_2(t, \overrightarrow{\boldsymbol{x}}), \qquad \boldsymbol{u}_2(t, \overrightarrow{\boldsymbol{x}}) = \boldsymbol{y}_1(t, \overrightarrow{\boldsymbol{x}}), \quad \forall\, t > 0,\ \overrightarrow{\boldsymbol{x}} \in \Gamma_{\text{int}}.$$

The **total Hamiltonian** of the coupled Heat-Wave system is given by:

$$\mathcal{H}(T, \partial_t w, \nabla w) := \underbrace{\frac{1}{2}\int_{\Omega_1} \left|T(t, \overrightarrow{\boldsymbol{x}})\right|^2 \mathrm{d}\overrightarrow{\boldsymbol{x}}}_{\mathcal{H}_T} + \underbrace{\frac{1}{2}\int_{\Omega_2} \left|\partial_t w(t, \overrightarrow{\boldsymbol{x}})\right|^2 + \left\|\nabla w(t, \overrightarrow{\boldsymbol{x}})\right\|^2 \mathrm{d}\overrightarrow{\boldsymbol{x}}}_{\mathcal{H}_w}.$$

Dissipative Power Balance

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_T + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_w \\
&= -\int_{\Omega_1} |J_Q|^2 + \langle \boldsymbol{u}_1, \boldsymbol{y}_1 \rangle_{H^{-\frac{1}{2}}(\Gamma_{\text{int}}), H^{\frac{1}{2}}(\Gamma_{\text{int}})} + \langle \boldsymbol{y}_2, \boldsymbol{u}_2 \rangle_{H^{-\frac{1}{2}}(\Gamma_{\text{int}}), H^{\frac{1}{2}}(\Gamma_{\text{int}})} \\
&= -\int_{\Omega_1} |J_Q|^2 - \langle \boldsymbol{y}_2, \boldsymbol{y}_1 \rangle_{H^{-\frac{1}{2}}(\Gamma_{\text{int}}), H^{\frac{1}{2}}(\Gamma_{\text{int}})} + \langle \boldsymbol{y}_2, \boldsymbol{y}_1 \rangle_{H^{-\frac{1}{2}}(\Gamma_{\text{int}}), H^{\frac{1}{2}}(\Gamma_{\text{int}})} \\
&= -\int_{\Omega_1} |J_Q|^2 .
\end{aligned}$$

Outline

3 steps: Weak formulation – Stokes identity – FEM

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Weak formulation:
For all test functions v_T, $\overrightarrow{\boldsymbol{v}}_Q$ on Ω_1, v_w, $\overrightarrow{\boldsymbol{v}}_w$ on Ω_2 and v_∂ on Γ_{int}:

$$\textbf{Heat}: \quad \begin{cases} \langle \partial_t T, v_T \rangle_{L^2(\Omega_1)} = \langle -\mathrm{div}\,(J_Q)\,, v_T \rangle_{L^2(\Omega_1)} \,, \\ \left\langle J_Q, \overrightarrow{\boldsymbol{v}}_Q \right\rangle_{\mathbf{L}^2(\Omega_1)} = \left\langle -\nabla T, \overrightarrow{\boldsymbol{v}}_Q \right\rangle_{\mathbf{L}^2(\Omega_1)} \,, \\ \langle v_\partial, \boldsymbol{y}_1 \rangle_{\Gamma_{\text{int}}} = \langle v_\partial, T \rangle_{\Gamma_{\text{int}}} \,, \end{cases}$$

3 steps: Weak formulation – Stokes identity – FEM

Weak formulation:
For all test functions v_T, $\vec{\boldsymbol{v}}_Q$ on Ω_1, v_w, $\vec{\boldsymbol{v}}_w$ on Ω_2 and v_∂ on Γ_{int}:

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Wave:
$$\begin{cases} \langle \partial_t(\partial_t w), v_w \rangle_{L^2(\Omega_2)} = \langle \text{div}\,(\nabla w)\,, v_w \rangle_{L^2(\Omega_2)}\,, \\ \left\langle \partial_t(\nabla w), \vec{\boldsymbol{v}}_w \right\rangle_{\mathbf{L}^2(\Omega_2)} = \left\langle \nabla \partial_t w, \vec{\boldsymbol{v}}_w \right\rangle_{\mathbf{L}^2(\Omega_2)}\,, \\ \langle \boldsymbol{y}_2, v_\partial \rangle_{\Gamma_{\text{int}}} = \langle \nabla w \cdot n_2, v_\partial \rangle_{\Gamma_{\text{int}}}\,, \end{cases}$$

3 steps: Weak formulation – Stokes identity – FEM

Weak formulation:
For all test functions v_T, $\vec{\boldsymbol{v}}_Q$ on Ω_1, v_w, $\vec{\boldsymbol{v}}_w$ on Ω_2 and v_∂ on Γ_{int}:

Heat:
$$\begin{cases} \langle \partial_t T, v_T \rangle_{L^2(\Omega_1)} = \langle -\mathrm{div}\,(J_Q)\,, v_T \rangle_{L^2(\Omega_1)}\,, \\ \left\langle J_Q, \vec{\boldsymbol{v}}_Q \right\rangle_{\mathbf{L}^2(\Omega_1)} = \left\langle -\nabla T, \vec{\boldsymbol{v}}_Q \right\rangle_{\mathbf{L}^2(\Omega_1)}\,, \\ \langle v_\partial, \boldsymbol{y}_1 \rangle_{\Gamma_{\mathrm{int}}} = \langle v_\partial, T \rangle_{\Gamma_{\mathrm{int}}}\,, \end{cases}$$

Wave:
$$\begin{cases} \langle \partial_t(\partial_t w), v_w \rangle_{L^2(\Omega_2)} = \langle \mathrm{div}\,(\nabla w)\,, v_w \rangle_{L^2(\Omega_2)}\,, \\ \left\langle \partial_t(\nabla w), \vec{\boldsymbol{v}}_w \right\rangle_{\mathbf{L}^2(\Omega_2)} = \left\langle \nabla \partial_t w, \vec{\boldsymbol{v}}_w \right\rangle_{\mathbf{L}^2(\Omega_2)}\,, \\ \langle \boldsymbol{y}_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}} = \langle \nabla w \cdot n_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}}\,, \end{cases}$$

Gyrator:
$$\begin{cases} \langle \boldsymbol{u}_1, v_\partial \rangle_{\Gamma_{\mathrm{int}}} = -\langle \boldsymbol{y}_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}}\,, \\ \langle v_\partial, \boldsymbol{u}_2 \rangle_{\Gamma_{\mathrm{int}}} = \langle v_\partial, \boldsymbol{y}_1 \rangle_{\Gamma_{\mathrm{int}}}\,. \end{cases}$$

Stokes (Green) identity:

Heat:
$$\begin{cases} \langle \partial_t T, v_T \rangle_{L^2(\Omega_1)} = \langle J_Q, \nabla v_T \rangle_{L^2(\Omega_1)} - \left\langle \underbrace{J_Q \cdot n_1}_{\boldsymbol{u}_1}, v_\partial \right\rangle_{\Gamma_{\text{int}}} - \left\langle \underbrace{J_Q \cdot n_1}_{\boldsymbol{y}_T}, v_\partial \right\rangle_{\Gamma_1}, \\ \left\langle J_Q, \overrightarrow{\boldsymbol{v}}_Q \right\rangle_{\mathbf{L}^2(\Omega_1)} = \left\langle -\nabla T, \overrightarrow{\boldsymbol{v}}_Q \right\rangle_{\mathbf{L}^2(\Omega_1)}, \\ \langle v_\partial, \boldsymbol{y}_1 \rangle_{\Gamma_{\text{int}}} = \langle v_\partial, T \rangle_{\Gamma_{\text{int}}}, \end{cases}$$

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$$\begin{cases} \langle \partial_t(\partial_t w), v_w \rangle_{L^2(\Omega_2)} = \langle \mathrm{div}\left(\nabla w\right), v_w \rangle_{L^2(\Omega_2)}, \\ \left\langle \partial_t(\nabla w), \overrightarrow{\boldsymbol{v}}_w \right\rangle_{\mathbf{L}^2(\Omega_2)} = \left\langle \partial_t w, -\mathrm{div}\left(\overrightarrow{\boldsymbol{v}}_w\right) \right\rangle_{\mathbf{L}^2(\Omega_2)} + \left\langle v_\partial, \underbrace{\partial_t w}_{\boldsymbol{u}_2} \right\rangle_{\Gamma_{\mathrm{int}}} + \left\langle v_\partial, \underbrace{\partial_t w}_{0} \right\rangle_{\Gamma_2}, \\ \langle \boldsymbol{y}_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}} = \langle \nabla w \cdot n_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}}, \end{cases}$$

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Heat:
$$\begin{cases} \langle \partial_t T, v_T \rangle_{L^2(\Omega_1)} = \langle J_Q, \nabla v_T \rangle_{L^2(\Omega_1)} - \left\langle \underbrace{J_Q \cdot n_1}_{\boldsymbol{u}_1}, v_\partial \right\rangle_{\Gamma_{\mathrm{int}}} - \left\langle \underbrace{J_Q \cdot n_1}_{\boldsymbol{y}_T}, v_\partial \right\rangle_{\Gamma_1}, \\ \left\langle J_Q, \overrightarrow{\boldsymbol{v}}_Q \right\rangle_{\mathbf{L}^2(\Omega_1)} = \left\langle -\nabla T, \overrightarrow{\boldsymbol{v}}_Q \right\rangle_{\mathbf{L}^2(\Omega_1)}, \\ \langle v_\partial, \boldsymbol{y}_1 \rangle_{\Gamma_{\mathrm{int}}} = \langle v_\partial, T \rangle_{\Gamma_{\mathrm{int}}}, \end{cases}$$

Wave:
$$\begin{cases} \langle \partial_t(\partial_t w), v_w \rangle_{L^2(\Omega_2)} = \langle \mathrm{div}\,(\nabla w)\,, v_w \rangle_{L^2(\Omega_2)}, \\ \left\langle \partial_t(\nabla w), \overrightarrow{\boldsymbol{v}}_w \right\rangle_{\mathbf{L}^2(\Omega_2)} = \left\langle \partial_t w, -\mathrm{div}\left(\overrightarrow{\boldsymbol{v}}_w\right) \right\rangle_{\mathbf{L}^2(\Omega_2)} + \left\langle v_\partial, \underbrace{\partial_t w}_{\boldsymbol{u}_2} \right\rangle_{\Gamma_{\mathrm{int}}} + \left\langle v_\partial, \underbrace{\partial_t w}_{0} \right\rangle_{\Gamma_2}, \\ \langle \boldsymbol{y}_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}} = \langle \nabla w \cdot n_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}}, \end{cases}$$

Gyrator:
$$\begin{cases} \langle \boldsymbol{u}_1, v_\partial \rangle_{\Gamma_{\mathrm{int}}} = -\langle \boldsymbol{y}_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}}, \\ \langle v_\partial, \boldsymbol{u}_2 \rangle_{\Gamma_{\mathrm{int}}} = \langle v_\partial, \boldsymbol{y}_1 \rangle_{\Gamma_{\mathrm{int}}}. \end{cases}$$

Stokes (Green) identity:

Heat:
$$\begin{cases} \langle \partial_t T, v_T \rangle_{L^2(\Omega_1)} = \langle J_Q, \nabla v_T \rangle_{L^2(\Omega_1)} - \left\langle \underbrace{J_Q \cdot n_1}_{\boldsymbol{u}_1}, v_\partial \right\rangle_{\Gamma_{\mathrm{int}}} - \left\langle \underbrace{J_Q \cdot n_1}_{\boldsymbol{y}_T}, v_\partial \right\rangle_{\Gamma_1}, \\ \langle J_Q, \overrightarrow{\boldsymbol{v}}_Q \rangle_{\mathbf{L}^2(\Omega_1)} = \langle -\nabla T, \overrightarrow{\boldsymbol{v}}_Q \rangle_{\mathbf{L}^2(\Omega_1)}, \\ \langle v_\partial, \boldsymbol{y}_1 \rangle_{\Gamma_{\mathrm{int}}} = \langle v_\partial, T \rangle_{\Gamma_{\mathrm{int}}}, \end{cases}$$

Warning: $T \equiv 0$ on Γ_1.

Wave:
$$\begin{cases} \langle \partial_t(\partial_t w), v_w \rangle_{L^2(\Omega_2)} = \langle \mathrm{div}\left(\nabla w\right), v_w \rangle_{L^2(\Omega_2)}, \\ \langle \partial_t(\nabla w), \overrightarrow{\boldsymbol{v}}_w \rangle_{\mathbf{L}^2(\Omega_2)} = \langle \partial_t w, -\mathrm{div}\left(\overrightarrow{\boldsymbol{v}}_w\right) \rangle_{\mathbf{L}^2(\Omega_2)} + \left\langle v_\partial, \underbrace{\partial_t w}_{\boldsymbol{u}_2} \right\rangle_{\Gamma_{\mathrm{int}}} + \left\langle v_\partial, \underbrace{\partial_t w}_{0} \right\rangle_{\Gamma_2}, \\ \langle \boldsymbol{y}_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}} = \langle \nabla w \cdot n_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}}, \end{cases}$$

Gyrator:
$$\begin{cases} \langle \boldsymbol{u}_1, v_\partial \rangle_{\Gamma_{\mathrm{int}}} = -\langle \boldsymbol{y}_2, v_\partial \rangle_{\Gamma_{\mathrm{int}}}, \\ \langle v_\partial, \boldsymbol{u}_2 \rangle_{\Gamma_{\mathrm{int}}} = \langle v_\partial, \boldsymbol{y}_1 \rangle_{\Gamma_{\mathrm{int}}}. \end{cases}$$

Projection on finite element basis: $(\xi_k^{\mathrm{int}})_{1\le k\le N_{\Gamma_{\mathrm{int}}}}$ for $\boldsymbol{u}_1$ & $\boldsymbol{u}_2$ & $\boldsymbol{y}_1$ & $\boldsymbol{y}_2$ and

$(\varphi_j^1)_{1\le j\le N_T}$	$(\overrightarrow{\psi}_i^1)_{1\le i\le N_Q}$	$(\xi_k^1)_{1\le k\le N_{\Gamma_1}}$	$(\varphi_j^2)_{1\le j\le N_p}$	$(\overrightarrow{\psi}_i^2)_{1\le i\le N_q}$	$(\xi_k^2)_{1\le k\le N_{\Gamma_2}}$
T	J_Q	0 & $\boldsymbol{y}_T$	$\partial_t w$	∇w	0 & $\boldsymbol{y}_w$

Projection on finite element basis: $(\xi_k^{\text{int}})_{1\leq k\leq N_{\Gamma_{\text{int}}}}$ for $\boldsymbol{u}_1$ & $\boldsymbol{u}_2$ & $\boldsymbol{y}_1$ & $\boldsymbol{y}_2$ and

$$\underset{T}{(\varphi_j^1)_{1\leq j\leq N_T}} \quad \underset{J_Q}{(\overrightarrow{\boldsymbol{\psi}}_i^1)_{1\leq i\leq N_Q}} \quad \underset{0\ \&\ \boldsymbol{y}_T}{(\xi_k^1)_{1\leq k\leq N_{\Gamma_1}}} \quad \underset{\partial_t w}{(\varphi_j^2)_{1\leq j\leq N_p}} \quad \underset{\nabla w}{(\overrightarrow{\boldsymbol{\psi}}_i^2)_{1\leq i\leq N_q}} \quad \underset{0\ \&\ \boldsymbol{y}_w}{(\xi_k^2)_{1\leq k\leq N_{\Gamma_2}}}$$

Heat:
$$\begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & \overrightarrow{\boldsymbol{M}}_1 & 0 & 0 \\ 0 & 0 & M_{bnd,1} & 0 \\ 0 & 0 & 0 & M_{\text{int}} \end{bmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{dt}}\underline{T} \\ \underline{J_Q} \\ \underline{0} \\ -\underline{y_1} \end{pmatrix} = \begin{bmatrix} 0 & D_1 & B_1 & B_{1,\text{int}} \\ -D_1^\top & 0 & 0 & 0 \\ -B_1^\top & 0 & 0 & 0 \\ -B_{1,\text{int}}^\top & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{T} \\ \underline{J_Q} \\ \underline{y_T} \\ \underline{u_1} \end{pmatrix},$$

$$(D_1)_{j,i} = \int_{\Omega_1} \overrightarrow{\boldsymbol{\psi}}_i^1 \cdot \nabla\varphi_j^1 \in \mathbb{R}^{N_T\times N_Q}, \quad (B_1)_{j,k} = -\int_{\Gamma_1} \xi_k^1\, \gamma_0(\varphi_j^1) \in \mathbb{R}^{N_T\times N_{\Gamma_1}},\ B_{1,int} \in \mathbb{R}^{N_T\times N_{\Gamma_{\text{int}}}}.$$

Partitioned Finite Element Method (PFEM)

Projection on finite element basis: $(\xi_k^{\text{int}})_{1\leq k\leq N_{\Gamma_{\text{int}}}}$ for $\boldsymbol{u}_1$ & $\boldsymbol{u}_2$ & $\boldsymbol{y}_1$ & $\boldsymbol{y}_2$ and

$$\underset{T}{(\varphi_j^1)_{1\leq j\leq N_T}} \quad \underset{J_Q}{(\vec{\boldsymbol{\psi}}_i^1)_{1\leq i\leq N_Q}} \quad \underset{0\ \&\ \boldsymbol{y}_T}{(\xi_k^1)_{1\leq k\leq N_{\Gamma_1}}} \quad \underset{\partial_t w}{(\varphi_j^2)_{1\leq j\leq N_p}} \quad \underset{\nabla w}{(\vec{\boldsymbol{\psi}}_i^2)_{1\leq i\leq N_q}} \quad \underset{0\ \&\ \boldsymbol{y}_w}{(\xi_k^2)_{1\leq k\leq N_{\Gamma_2}}}$$

Heat:

$$\begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & \vec{\boldsymbol{M}}_1 & 0 & 0 \\ 0 & 0 & M_{bnd,1} & 0 \\ 0 & 0 & 0 & M_{\text{int}} \end{bmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{T} \\ \underline{J_Q} \\ \underline{0} \\ -\underline{y_1} \end{pmatrix} = \begin{bmatrix} 0 & D_1 & B_1 & B_{1,\text{int}} \\ -D_1^\top & 0 & 0 & 0 \\ -B_1^\top & 0 & 0 & 0 \\ -B_{1,\text{int}}^\top & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{T} \\ \underline{J_Q} \\ \underline{y_T} \\ \underline{u_1} \end{pmatrix},$$

$$(D_1)_{j,i} = \int_{\Omega_1} \vec{\boldsymbol{\psi}}_i^1 \cdot \nabla \varphi_j^1 \in \mathbb{R}^{N_T\times N_Q}, \quad (B_1)_{j,k} = -\int_{\Gamma_1} \xi_k^1\, \gamma_0(\varphi_j^1) \in \mathbb{R}^{N_T\times N_{\Gamma_1}}, \; B_{1,int} \in \mathbb{R}^{N_T\times N_{\Gamma_{\text{int}}}}.$$

Wave:

$$\begin{bmatrix} M_2 & 0 & 0 & 0 \\ 0 & \vec{\boldsymbol{M}}_2 & 0 & 0 \\ 0 & 0 & M_{bnd,2} & 0 \\ 0 & 0 & 0 & M_{\text{int}} \end{bmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\partial_t w} \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\nabla w} \\ -\underline{y_w} \\ -\underline{y_2} \end{pmatrix} = \begin{bmatrix} 0 & D_2 & 0 & 0 \\ -D_2^\top & 0 & B_2 & B_{2,\text{int}} \\ 0 & -B_2^\top & 0 & 0 \\ 0 & -B_{2,\text{int}}^\top & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{\partial_t w} \\ \underline{\nabla w} \\ \underline{0} \\ \underline{u_2} \end{pmatrix},$$

$$(D_2)_{j,i} = \int_{\Omega_2} \mathrm{div}(\vec{\boldsymbol{\psi}}_i^2)\varphi_j^2 \in \mathbb{R}^{N_p\times N_q}, \quad (B_2)_{j,k} = \int_{\Gamma_2} \xi_k^2\, \vec{\boldsymbol{\psi}}_j^2) \cdot n_2 \in \mathbb{R}^{N_q\times N_{\Gamma_2}}, \; B_{2,int} \in \mathbb{R}^{N_q\times N_{\Gamma_{\text{int}}}}.$$

Projection on finite element basis: $(\xi_k^{\text{int}})_{1\le k\le N_{\Gamma_{\text{int}}}}$ for $\boldsymbol{u}_1$ & $\boldsymbol{u}_2$ & $\boldsymbol{y}_1$ & $\boldsymbol{y}_2$ and

$(\varphi_j^1)_{1\le j\le N_T}$	$(\vec{\boldsymbol{\psi}}_i^1)_{1\le i\le N_Q}$	$(\xi_k^1)_{1\le k\le N_{\Gamma_1}}$	$(\varphi_j^2)_{1\le j\le N_p}$	$(\vec{\boldsymbol{\psi}}_i^2)_{1\le i\le N_q}$	$(\xi_k^2)_{1\le k\le N_{\Gamma_2}}$
T	J_Q	0 & $\boldsymbol{y}_T$	$\partial_t w$	∇w	0 & $\boldsymbol{y}_w$

Heat:
$$\begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & \vec{\boldsymbol{M}}_1 & 0 & 0 \\ 0 & 0 & M_{bnd,1} & 0 \\ 0 & 0 & 0 & M_{\text{int}} \end{bmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{T} \\ \underline{J_Q} \\ \underline{0} \\ -\underline{y_1} \end{pmatrix} = \begin{bmatrix} 0 & D_1 & B_1 & B_{1,\text{int}} \\ -D_1^\top & 0 & 0 & 0 \\ -B_1^\top & 0 & 0 & 0 \\ -B_{1,\text{int}}^\top & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{T} \\ \underline{J_Q} \\ \underline{y_T} \\ \underline{u_1} \end{pmatrix},$$

$$(D_1)_{j,i} = \int_{\Omega_1} \vec{\boldsymbol{\psi}}_i^1 \cdot \nabla \varphi_j^1 \in \mathbb{R}^{N_T\times N_Q}, \quad (B_1)_{j,k} = -\int_{\Gamma_1} \xi_k^1\, \gamma_0(\varphi_j^1) \in \mathbb{R}^{N_T\times N_{\Gamma_1}}, \; B_{1,int} \in \mathbb{R}^{N_T\times N_{\Gamma_{\text{int}}}}.$$

Wave:
$$\begin{bmatrix} M_2 & 0 & 0 & 0 \\ 0 & \vec{\boldsymbol{M}}_2 & 0 & 0 \\ 0 & 0 & M_{bnd,2} & 0 \\ 0 & 0 & 0 & M_{\text{int}} \end{bmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\partial_t w} \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\nabla w} \\ -\underline{y_w} \\ -\underline{y_2} \end{pmatrix} = \begin{bmatrix} 0 & D_2 & 0 & 0 \\ -D_2^\top & 0 & B_2 & B_{2,\text{int}} \\ 0 & -B_2^\top & 0 & 0 \\ 0 & -B_{2,\text{int}}^\top & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{\partial_t w} \\ \underline{\nabla w} \\ \underline{0} \\ \underline{u_2} \end{pmatrix},$$

$$(D_2)_{j,i} = \int_{\Omega_2} \mathrm{div}(\vec{\boldsymbol{\psi}}_i^2)\varphi_j^2 \in \mathbb{R}^{N_p\times N_q}, \quad (B_2)_{j,k} = \int_{\Gamma_2} \xi_k^2\, \vec{\boldsymbol{\psi}}_j^2) \cdot n_2 \in \mathbb{R}^{N_q\times N_{\Gamma_2}}, \; B_{2,int} \in \mathbb{R}^{N_q\times N_{\Gamma_{\text{int}}}}.$$

Gyrator: $M_{\text{int}}\, \underline{u_1} = -M_{\text{int}}\, \underline{y_2}, \qquad M_{\text{int}}\, \underline{u_2} = M_{\text{int}}\, \underline{y_1}.$

Let $C := B_{1,\text{int}} M_{\text{int}}^{-1} B_{2,\text{int}}^\top$ the **gyrator interconnection** matrix.

$$\text{Diag}\begin{bmatrix} M_1 \\ \overrightarrow{\boldsymbol{M}}_1 \\ M_2 \\ \overrightarrow{\boldsymbol{M}}_2 \\ M_{bnd,1} \\ M_{bnd,2} \end{bmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{T} \\ \underline{J_Q} \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\partial_t w} \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\nabla w} \\ \underline{0} \\ -\underline{y_w} \end{pmatrix} = \begin{bmatrix} 0 & D_1 & 0 & -C & B_1 & 0 \\ -D_1^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_2 & 0 & 0 \\ C^\top & 0 & -D_2^\top & 0 & 0 & B_2 \\ -B_1^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2^\top & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{T} \\ \underline{J_Q} \\ \underline{\partial_t w} \\ \underline{\nabla w} \\ \underline{y_T} \\ \underline{0} \end{pmatrix}$$

Let $C := B_{1,\text{int}} M_{\text{int}}^{-1} B_{2,\text{int}}^\top$ the **gyrator interconnection** matrix.

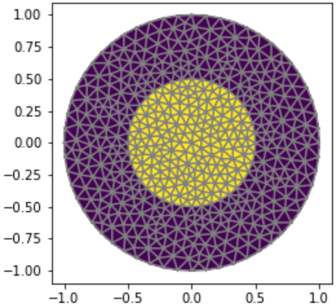
$$\text{Diag}\begin{bmatrix} M_1 \\ \overrightarrow{\boldsymbol{M}}_1 \\ M_2 \\ \overrightarrow{\boldsymbol{M}}_2 \\ M_{bnd,1} \\ M_{bnd,2} \end{bmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{T} \\ \underline{J_Q} \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\partial_t w} \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\nabla w} \\ \underline{0} \\ -\underline{y_w} \end{pmatrix} = \begin{bmatrix} 0 & D_1 & 0 & -C & B_1 & 0 \\ -D_1^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_2 & 0 & 0 \\ C^\top & 0 & -D_2^\top & 0 & 0 & B_2 \\ -B_1^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2^\top & 0 & 0 \end{bmatrix} \begin{pmatrix} \underline{T} \\ \underline{J_Q} \\ \underline{\partial_t w} \\ \underline{\nabla w} \\ \underline{y_T} \\ \underline{0} \end{pmatrix}$$

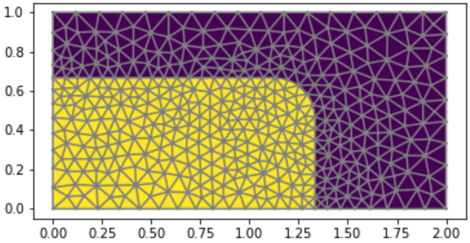
Discrete Lossy Power Balance

The discrete Hamiltonian $\mathcal{H}^d$ is defined as the continuous Hamiltonian $\mathcal{H}$ evaluated in the *approximated solution*.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^d(\underline{T}, \underline{\partial_t w}, \underline{\nabla w}) = -\underline{J_Q}^\top \overrightarrow{\boldsymbol{M}}_1 \underline{J_Q}.$$

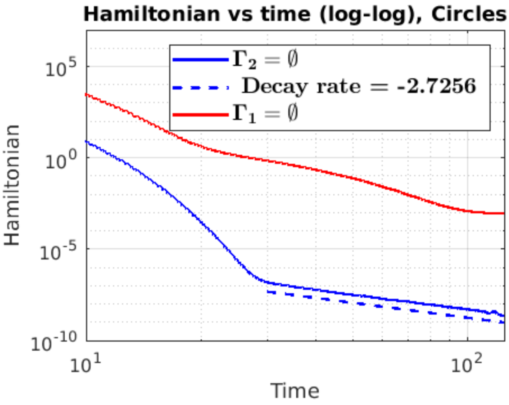
Outline

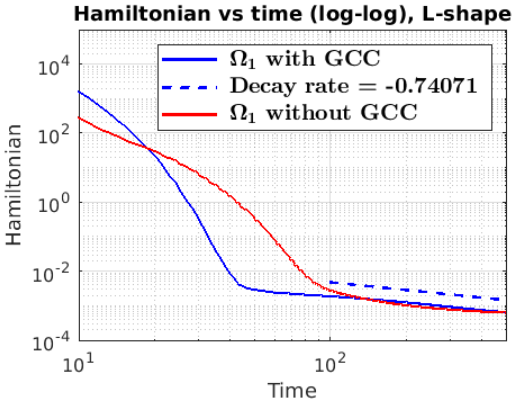




Switching Ω_1 and Ω_2, **four cases** are covered.

Hamiltonian decays





GCC holds, $\Gamma_1 \neq \emptyset$ and $\Gamma_2 = \emptyset$

GCC fails, $\Gamma_1 = \emptyset$ and $\Gamma_2 \neq \emptyset$

GCC holds, $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$

GCC fails, $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$

Outline

Conclusion and Further Works

To sum up:

A numerical validation of the interest of PHS has been performed:

- The long-time behaviour is as expected at the discrete level;
- The simulation process is intrinsically object oriented.

To go further:

- Choice for the **finite elements families**?
- **Symplectic** time scheme? ⚠ *DAE*!!!
- Structure-preserving **model reduction**?

References

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Thank you for your attention!

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