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Modelling the 1D piston problem as interconnected port-Hamiltonian systems

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Outline

1. Introduction
   - The 1D piston problem
   - Physics
   - Port-Hamiltonian systems
2. Modelling the 1D piston
   - Kinetic energy
   - Internal energy
   - One sub-system
   - The 1D piston problem
3. Conclusion

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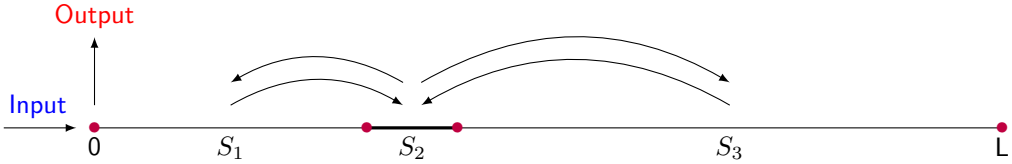

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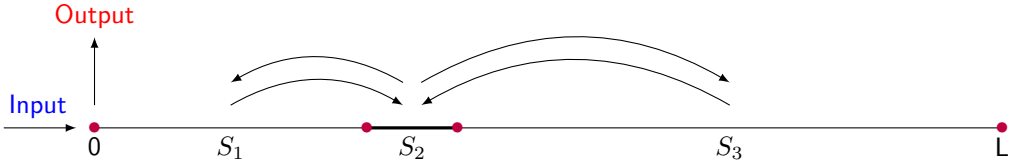

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Notations:

ρ: mass density;
σ: stress tensor;
s: entropy density;
π: local pressure;
v: particle velocity;
u: internal energy density;
T: local temperature;
V: volume (length);
$p := \rho v$: linear momentum;
J_q: heat flux;
$\beta := \frac{1}{T}$: reciprocal temperature;
$J_s := \frac{1}{T} J_q$: entropy flux;

$\Sigma := \frac{1}{T}\left(\sigma \partial_x v - J_s \partial_x T + \pi \frac{\mathrm{d}}{\mathrm{d}t} V\right)$: irreversible entropy production.

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Axiomatics:

- Conservation of mass: $\partial_t \rho = -\partial_x(\rho v)$;
- Conservation of linear momentum: $\partial_t p = -\partial_x(pv) + \partial_x(\sigma)$;
- Conservation of internal energy: $\partial_t(\bar{u}) = -\partial_x(\bar{u}v) + \sigma\partial_x(v) - \partial_x(J_q), \qquad \bar{u} := \rho u$;
- Gibb's formula: $\mathrm{d}\bar{u} = T\ \mathrm{d}\bar{s} - \pi\ \mathrm{d}V + \sum_i \mu_i\ \mathrm{d}n_i, \qquad \bar{s} := \rho s.$

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Context & comportemental laws (equations of state):

- 1D domain: $V := (a, b)$;
- No chemical reactions is to be found in the system (inert media): $\mu_i \equiv 0$;
- Dulong-Petit model: $u = C_v T, \qquad C_v$ isochoric heat capacity.
- Fourier's law for heat condution: $J_q = -k\nabla(T)$;
- For fluids S_1, S_3, stress tensors are of the form: $\nu\partial_x v - \pi$;
- Ideal gas law: $\pi V = nRT$, where n is the amount of substance and R the universal gas constant;
- For the rigid S_2, the stress tensor is: $\sigma \equiv 0$ (Hooke's law with null elasticity modulus).

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Lossy Power Balance: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\boldsymbol{\alpha}(t)) = -\left\langle R\boldsymbol{e_\alpha}(t), \boldsymbol{e_\alpha}(t)\right\rangle_J + \left\langle \boldsymbol{u}(t), \boldsymbol{y}(t)\right\rangle_B \leq \left\langle \boldsymbol{u}(t), \boldsymbol{y}(t)\right\rangle_B .$

/!\ Although **the underlying geometry** is well-determined with the above equation,
constitutive relations between $\boldsymbol{\alpha}$ and $\boldsymbol{e_\alpha}$ are also needed to solve the system!

Port-Hamiltonian systems

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- The **Bond space** $\mathcal{B} := \mathcal{F} \times \mathcal{E}$, with symmetrized bilinear product:

$$\left[\begin{pmatrix} \boldsymbol{f}^1 \\ \boldsymbol{e}^1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{f}^2 \\ \boldsymbol{e}^2 \end{pmatrix} \right]_{\mathcal{B}} := \left\langle \boldsymbol{f}^1, \boldsymbol{e}^2 \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \boldsymbol{f}^2, \boldsymbol{e}^1 \right\rangle_{\mathcal{F},\mathcal{E}};$$

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Theorem

If $\mathcal{J} \in \mathcal{L}(\mathcal{E}_1, \mathcal{F})$ is closed, densely defined and: $\left\langle \mathcal{J} \boldsymbol{e}^1, \boldsymbol{e}^2 \right\rangle_{\mathcal{F},\mathcal{E}} = - \left\langle \mathcal{J} \boldsymbol{e}^2, \boldsymbol{e}^1 \right\rangle_{\mathcal{F},\mathcal{E}}$ for all $\boldsymbol{e}_1, \boldsymbol{e}_2 \in \mathcal{E}_1$, then $\mathcal{D} := \mathrm{Graph}\,(\mathcal{J}) \subset \mathcal{B}$ is a **Dirac structure**, *i.e.* $\mathcal{D}^{[\perp]} = \mathcal{D}$ with:

$$\mathcal{D}^{[\perp]} := \left\{ \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{e} \end{pmatrix} \in \mathcal{B} \;\middle|\; \left[\begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{e} \end{pmatrix}, \begin{pmatrix} \widetilde{\boldsymbol{f}} \\ \widetilde{\boldsymbol{e}} \end{pmatrix} \right]_{\mathcal{B}} = 0, \quad \forall \begin{pmatrix} \widetilde{\boldsymbol{f}} \\ \widetilde{\boldsymbol{e}} \end{pmatrix} \in \mathcal{D} \right\}.$$

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$$\langle \boldsymbol{f}(t), \boldsymbol{e}(t) \rangle_{\mathcal{F},\mathcal{E}} = 0, \quad \forall \left(\boldsymbol{f}(t), \boldsymbol{e}(t) \right) \in \mathcal{D}, \quad \forall t \geq 0.$$

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Let $(\partial_t \boldsymbol{\alpha}, \boldsymbol{f}_R, -\boldsymbol{y}, \boldsymbol{e}_{\boldsymbol{\alpha}}, \boldsymbol{e}_R, \boldsymbol{u})^\top$ be in $\mathcal{D}$. Adding $\boldsymbol{e}_R = R\boldsymbol{f}_R$: **the lossy power balance is satisfied**!

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Main result

- An **underlying Dirac structure** *containing* the **power balance** is constructed for both the **kinetic** and **internal energies** of S_1, S_2 and S_3.
- The **six** resulting systems are interconnected to obtain a direct modelling of the 1D piston problem as a port-Hamiltonian system with algebraic constraints.
- An interesting feature is that this modelling keeps the **geometric structure of conservation laws** apart from the comportemental laws: the **constitutive relations**.

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Kinetic energy

Kinetic energy as Hamiltonian: $\mathcal{H}_k(p,\rho) := \frac{1}{2}\int_{a(t)}^{b(t)} \frac{p^2(t,x)}{\rho(t,x)}\mathrm{d}x$, with p and ρ as energy variables.

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Power balance: From the transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_k(p,\rho) = \underbrace{[\rho(t,x)v(t,x)]_{a(t)}^{b(t)}}_{\text{Injection–rejection}} - \underbrace{\left[\rho(t,x)\frac{\mathrm{d}}{\mathrm{d}t}x\right]_{a(t)}^{b(t)}}_{\text{Moving boundary}} + \underbrace{[v(t,x)\sigma(t,x)]_{a(t)}^{b(t)}}_{\text{Force applied}} - \underbrace{\int_{a(t)}^{b(t)} \partial_x v(t,x)\sigma(t,x)\mathrm{d}x}_{\text{Loss in thermal domain}}.$$

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This rewrites: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_k(p,\rho) = \langle \boldsymbol{u}_r, \boldsymbol{y}_r\rangle_{\mathbb{R}^2} + \langle \boldsymbol{u}_\sigma, \boldsymbol{y}_\sigma\rangle_{\mathbb{R}^2} - \langle \partial_x v, \sigma\rangle_{L^2(a(t),b(t))}$,

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$\boldsymbol{u}_\sigma(t) := \begin{pmatrix} -\sigma(t,a(t)), & \sigma(t,b(t)) \end{pmatrix}^\top$, $\qquad \boldsymbol{y}_\sigma(t) := \gamma_0\left(v(t,x)\right) = \begin{pmatrix} v(t,a(t)), & v(t,b(t)) \end{pmatrix}^\top$,

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$\boldsymbol{y}_r(t) := \gamma_0\left(-\frac{v^2(t,x)}{2}\right) = \left(-\frac{v^2(t,a(t))}{2},\quad -\frac{v^2(t,b(t))}{2}\right)^\top$.

Kinetic energy

Flows and efforts:

$$\boldsymbol{f}_k := \left(\partial_t p, \quad \partial_t \rho, \quad -\partial_x v, \quad \partial_x v, \quad \partial_x \left(\frac{v^2}{2}\right), \quad -\boldsymbol{y}_\sigma, \quad -\boldsymbol{y}_r\right)^\top,$$

$$\boldsymbol{e}_k := \left(\quad v, \quad -\frac{v^2}{2}, \quad pv, \quad \sigma, \quad \rho v, \quad \boldsymbol{u}_\sigma, \quad \boldsymbol{u}_r\right)^\top.$$

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Extended structure operator:

$$\mathcal{J}_k := \begin{pmatrix} 0 & 0 & -\partial_x & \partial_x & 0 & \gamma_0^* & 0 \\ 0 & 0 & 0 & 0 & -\partial_x & 0 & \gamma_0^* \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 & 0 & 0 & 0 \\ -\gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\gamma_0^* \in \mathcal{L}\left(\mathbb{R}^2, H^{-1}(a,b)\right)$, adjoint of γ_0,

$H^{-1}(a,b)$ dual of $H_0^1(a,b) := \ker(\gamma_0)$, w.r.t. $L^2(a,b)$.

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$$\mathcal{J}_k := \begin{pmatrix} 0 & 0 & -\partial_x & \partial_x & 0 & \gamma_0^* & 0 \\ 0 & 0 & 0 & 0 & -\partial_x & 0 & \gamma_0^* \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 & 0 & 0 & 0 \\ -\gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\gamma_0^* \in \mathcal{L}\left(\mathbb{R}^2, H^{-1}(a,b)\right)$, adjoint of γ_0,

$H^{-1}(a,b)$ dual of $H_0^1(a,b) := \ker(\gamma_0)$, w.r.t. $L^2(a,b)$.

Theorem

Let $\mathcal{E}_1^k := \left(H^1(a(t),b(t))\right)^5 \times \left(\mathbb{R}^2\right)^2$,

$\mathcal{E}^k := \left(L^2(a(t),b(t))\right)^5 \times \left(\mathbb{R}^2\right)^2$, $\quad \mathcal{F}^k := \left(\mathcal{E}^k\right)' = \left(L^2(a(t),b(t))\right)^5 \times \left(\mathbb{R}^2\right)^2$, $\quad \mathcal{B}^k := \mathcal{F}^k \times \mathcal{E}^k$,

then $\mathcal{D}_k := \mathrm{Graph}(\mathcal{J}_k)$ is an **underlying Dirac structure** of $\mathcal{H}_k$ on $\mathcal{B}^k$.

Outline

1 Introduction

2 Modelling the 1D piston
- Kinetic energy
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3 Conclusion

Internal energy

Internal energy as Hamiltonian: $\mathcal{H}_u(\overline{s}, \rho) := \int_{a(t)}^{b(t)} \overline{u}(\overline{s}(t,x))\mathrm{d}x$, with $\overline{s}$ and ρ as energy variables.

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Co-energy variables: $e_{\overline{s}} := \delta_{\overline{s}}\mathcal{H}_u = T$, the *local temperature*,
$e_\rho := \delta_\rho \mathcal{H}_u = u$, the *internal energy density*.

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Power balance: From the transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_u(\overline{s},\rho) = \underbrace{\left[\overline{u}(t,x)\frac{\mathrm{d}}{\mathrm{d}t}x\right]_{a(t)}^{b(t)}}_{\text{Moving boundary}} - \underbrace{[\overline{u}(t,x)v(t,x)]_{a(t)}^{b(t)}}_{\text{Injection–rejection}} - \underbrace{[J_q(t,x)]_{a(t)}^{b(t)}}_{\text{Heating–cooling}} + \underbrace{\int_{a(t)}^{b(t)} \partial_x v(t,x)\sigma(t,x)\mathrm{d}x}_{\text{Gain from kinetic domain}}.$$

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This rewrites: $$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_u(\overline{s},\rho) = -\langle \widetilde{\boldsymbol{u}}_r, \widetilde{\boldsymbol{y}}_r\rangle_{\mathbb{R}^2} + \langle \boldsymbol{u}_s, \boldsymbol{y}_s\rangle_{\mathbb{R}^2} + \langle \partial_x v, \sigma\rangle_{L^2(a(t),b(t))} ,$$

Internal energy

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$\boldsymbol{u}_s(t) := \gamma_0\left(T(t,x)\right) = \begin{pmatrix} T(t,a(t)), & T(t,b(t)) \end{pmatrix}^\top,$ $\qquad \boldsymbol{y}_s(t) := \begin{pmatrix} J_s(t,a(t)), & -J_s(t,b(t)) \end{pmatrix}^\top,$

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This rewrites: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_u(\overline{s},\rho) = -\langle \widetilde{\boldsymbol{u}}_r, \widetilde{\boldsymbol{y}}_r\rangle_{\mathbb{R}^2} + \langle \boldsymbol{u}_s, \boldsymbol{y}_s\rangle_{\mathbb{R}^2} + \langle \partial_x v, \sigma\rangle_{L^2(a(t),b(t))},$

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$\widetilde{\boldsymbol{u}}_r(t) := \begin{pmatrix} \rho(t,a(t))v(t,a(t)) - \rho(t,a(t))\frac{\mathrm{d}}{\mathrm{d}t}a(t), & -\rho(t,b(t))v(t,b(t)) + \rho(t,b(t))\frac{\mathrm{d}}{\mathrm{d}t}b(t)\end{pmatrix}^\top = -\boldsymbol{u}_r(t)$,

$\widetilde{\boldsymbol{y}}_r(t) := \gamma_0\left(u\right) = \begin{pmatrix} u(t,a(t)), & u(t,b(t))\end{pmatrix}^\top$.

Internal energy

Flows and efforts:

$$\boldsymbol{f}_k := \big(\partial_t \overline{s},\quad \partial_t \rho,\quad -\partial_x T,\quad -\partial_x T,\quad -T,\quad -\partial_x u,\quad \boldsymbol{u}_s,\quad -\widetilde{\boldsymbol{y}}_r\big)^\top,$$
$$\boldsymbol{e}_k := \big(\ T,\quad u,\quad \overline{s}v,\quad J_s,\quad \Sigma,\quad \rho v,\quad -\boldsymbol{y}_s,\quad \widetilde{\boldsymbol{u}}_r\big)^\top.$$

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Extended structure operator:

$$\mathcal{J}_u := \begin{pmatrix} 0 & 0 & -\partial_x & -\partial_x & I & 0 & -\gamma_0^* & 0 \\ 0 & 0 & 0 & 0 & 0 & -\partial_x & 0 & \gamma_0^* \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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Theorem

Let $\mathcal{E}_1^u := \big(H^1(a(t),b(t))\big)^6 \times \big(\mathbb{R}^2\big)^2$,

$\mathcal{E}^u := \big(L^2(a(t),b(t))\big)^6 \times \big(\mathbb{R}^2\big)^2$, $\mathcal{F}^u := (\mathcal{E}^u)' = \big(L^2(a(t),b(t))\big)^6 \times \big(\mathbb{R}^2\big)^2$, $\mathcal{B}^u := \mathcal{F}^u \times \mathcal{E}^u$,

then $\mathcal{D}_u := \mathrm{Graph}(\mathcal{J}_u)$ is an **underlying Dirac structure** of $\mathcal{H}_u$ on $\mathcal{B}^u$.

Outline

1 Introduction

2 Modelling the 1D piston
- Kinetic energy
- Internal energy
- One sub-system
- The 1D piston problem

3 Conclusion

One sub-system

Interconnection on a sub-system: $\mathcal{H}_{S_i}(p, \overline{s}, \rho) := \mathcal{H}_k(p, \rho) + \mathcal{H}_u(\overline{s}, \rho)$ is the total energy in S_i.

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Power balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{S_i}(p, \overline{s}, \rho) = \langle \boldsymbol{u}_\sigma, \boldsymbol{y}_\sigma \rangle_{\mathbb{R}^2} + \langle \boldsymbol{u}_r, \boldsymbol{y}_r - \widetilde{\boldsymbol{y}}_r \rangle_{\mathbb{R}^2} + \langle \boldsymbol{u}_s, \boldsymbol{y}_s \rangle_{\mathbb{R}^2}\,, \quad \text{thanks to } \boldsymbol{u}_r = -\widetilde{\boldsymbol{u}}_r.$$

The total energy of the closed system is conservative, as expected.

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The total energy of the closed system is conservative, as expected.

Remark: $\boldsymbol{y}_r - \widetilde{\boldsymbol{y}}_r = \gamma_0\left(u + \frac{v^2}{2}\right)$ is the Dirichlet trace of the total energy density.

One sub-system

Extended structure operator on a $(a(t), b(t))$, for each fixed $t \geq 0$:

$\partial_t \rho = \partial_t \rho$ by Lagrange multiplier

$$\begin{pmatrix} \partial_t p \\ \partial_t \rho \\ -\partial_x v \\ \partial_x v \\ \partial_x \left(\frac{v^2}{2}\right) \\ -\boldsymbol{y_\sigma} \\ -\boldsymbol{y_r} \\ \partial_t \overline{s} \\ \partial_t \rho \\ -\partial_x T \\ -\partial_x T \\ -T \\ -\partial_x u \\ \boldsymbol{u_s} \\ -\widetilde{\boldsymbol{y}}_r \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 & -\partial_x & \partial_x & 0 & \gamma_0^* & 0 \\ 0 & 0 & 0 & 0 & -\partial_x & 0 & \gamma_0^* \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 & 0 & 0 & 0 \\ -\gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \mathbf{0} & \begin{pmatrix} 0 & 0 & 0 \\ I_{L^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -I_{L^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_{\mathbb{R}^2} \end{pmatrix} \\ \mathbf{0} & \begin{pmatrix} 0 & 0 & -\partial_x & -\partial_x & I & 0 & -\gamma_0^* & 0 \\ 0 & 0 & 0 & 0 & 0 & -\partial_x & 0 & \gamma_0^* \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ -I_{L^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_{\mathbb{R}^2} \end{pmatrix} \\ \begin{pmatrix} 0 & -I_{L^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{L^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathbb{R}^2} \end{pmatrix} & \mathbf{0} & \begin{pmatrix} 0 & I_{L^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -I_{L^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\mathbb{R}^2} \end{pmatrix} & \mathbf{0} \end{pmatrix} \begin{pmatrix} v \\ -\frac{v^2}{2} \\ pv \\ \sigma \\ \rho v \\ \boldsymbol{u_\sigma} \\ \boldsymbol{u_r} \\ T \\ u \\ \overline{s}v \\ J_s \\ \Sigma \\ \rho v \\ -\boldsymbol{y}_s \\ \widetilde{\boldsymbol{u}}_r \\ u + \frac{v^2}{2} \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \partial_t \rho = \partial_t \rho \\ \rho v = \rho v \\ \boldsymbol{u_r} = -\widetilde{\boldsymbol{u}}_r \end{matrix}$$

Remark: The Lagrange multiplier for $\partial_t \rho = \partial_t \rho$ is the total energy density $u + \frac{v^2}{2}$.

In orange: Sign errors in the proceeding.

Outline

1 Introduction

2 Modelling the 1D piston
- Kinetic energy
- Internal energy
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The 1D piston problem

Interconnection of S_1, S_2, and S_3 on $(a_0, a_3) = (0, L)$:

Assuming no matter exchange between sub-systems, neither between S_3 and the environment, and:

- thermodynamical equilibrium: $J_{q,1}(t, a_1(t)) = -J_{q,2}(t, a_1(t)), \quad J_{q,2}(t, a_2(t)) = -J_{q,3}(t, a_2(t));$
- mechanical equilibrium: $\sigma_1(t, a_1(t)) = -\sigma_2(t, a_1(t)), \quad \sigma_2(t, a_2(t)) = -\sigma_3(t, a_2(t));$
- internal boundary velocities only driven by the matter:

$$v_1(t, a_1(t)) = v_2(t, a_1(t)) = \mathrm{d}_t a_1(t), \quad v_2(t, a_2(t)) = v_3(t, a_2(t)) = \mathrm{d}_t a_2(t).$$

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Power balance: $\mathcal{H}_{\mathrm{Tot}} := \mathcal{H}_{S_1} + \mathcal{H}_{S_2} + \mathcal{H}_{S_3}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_{\mathrm{Tot}} = \underbrace{\rho_1(t,0)v_1(t,0)}_{\text{Injection–rejection}}\left(\frac{{v_1}^2(t,0)}{2} + u(t,0)\right) - \underbrace{\sigma_1(t,0)}_{\text{Pressure}} v_1(t,0) - \underbrace{J_{q,1}(t,0) + J_{q,3}(t,L)}_{\text{Heating–cooling}}.$$

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Extended structure operator:
It is constructed thanks to the above contextual assumptions by writing the interactions in terms of equalities between inputs $\boldsymbol{u}$ and ouputs $\boldsymbol{y}$.

The 1D piston problem

With the additional hypotheses (in order to close the system with constitutive relations):

- fluids S_1 and S_3 are ideal Newtonian gas: $\sigma_i = \nu_i \partial_x v_i - \frac{n_i R T_i}{V_i}$, $i = 1, 2$;
- solid S_2 is rigid: $\sigma_2 \equiv 0$;
- the heat transferts follow Fourier's law: $J_{q,i} = -\lambda_i \partial_x T_i$, $i = 1, 2, 3$;
- the Dulong-Petit model is valid: $\overline{u}_i = C_{v,i} \rho_i T_i$, $i = 1, 2, 3$;

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$$
\begin{cases}
\partial_t \rho_1 = -\partial_x(\rho_1 v_1), & (0, a_1),\\
\partial_t \rho_1 v_1 = -\partial_x(\rho_1 {v_1}^2) + \partial_x\left(\nu_1 \partial_x(v_1) - \frac{n_1 R T_1}{a_1}\right), & (0, a_1),\\
C_{v,1} \partial_t(\rho_1 T_1) = -\partial_x(C_{v,1} \rho_1 T_1) + \left(\nu_1 \partial_x(v_1) - \frac{n_1 R T_1}{a_1}\right) \partial_x(v_1) + \partial_x(\lambda_1 \partial_x(T_1)), & (0, a_1),\\
\rho_2 \partial_t v_2 = -(v_2)_2 \partial_x(\rho_2), & (a_1, a_2),\\
C_{v,2} \rho_2 \partial_t(T_2) = -\partial_x(C_{v,2} \rho_2 T_2) + \partial_x(\lambda_2 \partial_x(T_2)), & (a_1, a_2),\\
\partial_t \rho_3 = -\partial_x(\rho_3 v_3), & (a_2, L),\\
\partial_t \rho_3 v_3 = -\partial_x(\rho_3 {v_3}^2) + \partial_x\left(\nu_3 \partial_x(v_3) - \frac{n_3 R T_3}{L - a_2}\right), & (a_2, L),\\
C_{v,3} \partial_t(\rho_3 T_3) = -\partial_x(C_{v,3} \rho_3 T_3) + \left(\nu_3 \partial_x(v_3) - \frac{n_3 R T_3}{L - a_2}\right) \partial_x(v_3) + \partial_x(\lambda_3 \partial_x(T_3)), & (a_2, L),\\
\oplus \text{ boundary conditions.}
\end{cases}
$$

Remark: n^1 is a part of the injection–rejection control as it modifies the amount of substance in S_1!

Outline

1 Introduction

2 Modelling the 1D piston

3 Conclusion

Conclusion

We propose a new approach for the modelling of the 1D piston problem, as interconnected port-Hamiltonian systems. This allows:

- **meaningful** physical quantities;
- the construction of an algebraic structure encoding the power balance: a **Dirac structure**;
- the **postponment** of equations of state (Fourier's law, Newtonian fluids, ideal gas, rigid solid, Dulong-Petit model) at the end of the process.

Further works:

- structure-preserving discretization taking advantage of the knowledge of the Dirac structure;
- 2D and 3D models, with chemical reactions.

Thank you for your attention!

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