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Partitioned finite element method for structured discretization with mixed boundary conditions

Andrea Brugnoli[1], Flávio Luiz Cardoso-Ribeiro[2], Ghislain Haine[1], Paul Kotyczka[3]

[1]ISAE-SUPAERO, France

[2]Instituto Tecnológico de Aeronáutica, Brazil

[3]Technical University of Munich, Germany

Outline

Plan

Model description

Model for the propagation of sound in air

$$\frac{\partial}{\partial t}\begin{bmatrix}\chi_s p \\ \mu_0 \boldsymbol{v}\end{bmatrix} = -\begin{bmatrix}0 & \mathrm{div} \\ \mathrm{grad} & 0\end{bmatrix}\begin{bmatrix}p \\ \boldsymbol{v}\end{bmatrix}, \qquad \text{on } \Omega = \{x \in [0, L],\ r \in [0, R],\ \theta = [0, 2\pi)\}.$$

- $p \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^3$: variations of pressure and velocity from a steady state;
- μ_0: the steady state mass density;
- χ_s: adiabatic compressibility factor;
- x, r, θ: axial, radial and tangential coordinates.

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Boundary conditions

$$\begin{aligned} p(x, R, \theta) &= -\mathcal{Z}(x, t)\, v_r(x, R, \theta), \\ \boldsymbol{v} \cdot \boldsymbol{n}(0, r, \theta) &= -v_x(0, r, \theta) = -f(r), \\ \boldsymbol{v} \cdot \boldsymbol{n}(L, r, \theta) &= +v_x(L, r, \theta) = +f(r), \end{aligned}$$

Initial conditions

$$\begin{aligned} p^0(x, r, \theta) &= 0, & v_r^0(x, r, \theta) &= g(r), \\ v_x^0(x, r, \theta) &= f(r), & v_\theta^0(x, r, \theta) &= 0. \end{aligned}$$

The impedance $\mathcal{Z}$ and the axial $f(r)$ and radial flow $g(r)$ expressions are the following

$$\mathcal{Z}(x, t) = \mathbb{1}\left\{\frac{1}{3}L \leq \ x \ \leq \frac{2}{3}L,\ t \geq 0.2\ t_{\text{fin}}\right\} \mu_0\, c_0,$$

$$f(r) = \left(1 - \frac{r^2}{R^2}\right) v_0, \qquad g(r) = 16\frac{r^2}{R^4}(R - r)^2\, v_0.$$

Model for the propagation of sound in air

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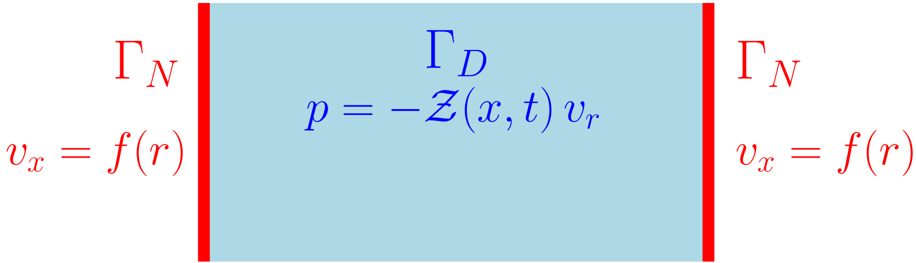

Model reduction by symmetry

Because of symmetry the model can be reduced to a 2D problem

$$\frac{\partial}{\partial t}\begin{bmatrix} \chi_s p \\ \mu_0 v_x \\ \mu_0 v_r \end{bmatrix} = -\begin{bmatrix} 0 & \partial_x & \partial_r + 1/r \\ \partial_x & 0 & 0 \\ \partial_r & 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ v_x \\ v_r \end{bmatrix}, \quad \text{on } \Omega_{\mathsf{r}} = \{x \in [0, L], r \in [0, R]\}.$$

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The boundary conditions must now account for the symmetry condition at $r = 0$

$$\begin{aligned} p(x, R, \theta) &= -\mathcal{Z}(x,t)\, v_r(x, R, \theta), \\ \boldsymbol{v} \cdot \boldsymbol{n}(0, r, \theta) &= -v_x(0, r, \theta) = -f(r), \\ \boldsymbol{v} \cdot \boldsymbol{n}(L, r, \theta) &= +v_x(L, r, \theta) = +f(r), \\ \boldsymbol{v} \cdot \boldsymbol{n}(x, 0) &= v_r(x, 0) = 0 \end{aligned}$$

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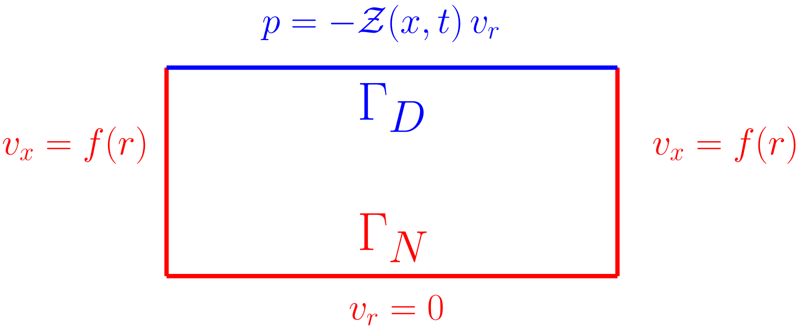
$$\frac{\partial}{\partial t}\begin{bmatrix}\chi_s p \\ \mu_0 v_x \\ \mu_0 v_r\end{bmatrix} = -\begin{bmatrix}0 & \partial_x & \partial_r + 1/r \\ \partial_x & 0 & 0 \\ \partial_r & 0 & 0\end{bmatrix}\begin{bmatrix}p \\ v_x \\ v_r\end{bmatrix}, \quad \text{on } \Omega_\mathrm{r} = \{x \in [0, L], r \in [0, R]\}.$$



A port-Hamiltonian structure

The system can be rewritten compactly as a pH system in co-energy variables

$$\mathcal{M}\partial_t e = \mathcal{J} e$$

where $\mathcal{M} = \mathrm{diag}([\chi_s,\ \mu_0,\ \mu_0])$ and $e = [e_p,\ \boldsymbol{e}_v]^\top = [p,\ \boldsymbol{v}]^\top$.

The Hamiltonian is then computed as

$$H = \frac{1}{2}(e, \mathcal{M}e)_{\Omega_\mathrm{r}}$$

where $(\cdot,\cdot)_{\Omega_\mathrm{r}}$ is the standard L^2 inner product in polar coordinates

$$(\alpha, \beta)_{\Omega_\mathrm{r}} = \int_{\Omega_\mathrm{r}} \alpha \cdot \beta \ r \,\mathrm{d}r\,\mathrm{d}x = \int_{\Omega_\mathrm{r}} \alpha \cdot \beta \ \ \mathrm{d}\Omega_r.$$

The power flow is obtained by application of the Stokes theorem

$$\dot{H} = -\int_0^L \mathcal{Z}(x,t) v_r^2 \ R \,\mathrm{d}x \leq 0$$

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where $\mathcal{M} = \mathrm{diag}([\chi_s, \ \mu_0, \ \mu_0])$ and $e = [e_p, \ \boldsymbol{e}_v]^\top = [p, \ \boldsymbol{v}]^\top$.

The interconnection operator $\mathcal{J}$ can be decomposed into the sum of $\mathcal{J} = \mathcal{J}_{\text{div}} + \mathcal{J}_{\text{grad}}$

$$\mathcal{J}_{\text{div}} = -\begin{bmatrix} 0 & \mathrm{div}_r \\ 0 & 0 \end{bmatrix}, \qquad \mathrm{div}_r = [\partial_x, \ \partial_r + 1/r]$$

$$\mathcal{J}_{\text{grad}} = -\begin{bmatrix} 0 & 0 \\ \mathrm{grad}_r & 0 \end{bmatrix}, \qquad \mathrm{grad}_r = \begin{pmatrix} \partial_x \\ \partial_r \end{pmatrix}.$$

Plan

The partitioned finite element method (PFEM)

General procedure for PFEM

1. Put the system into weak form:
$$\left(v, \mathcal{M}\frac{\partial e}{\partial t}\right)_\Omega = (v, \mathcal{J} e)_\Omega \, .$$
2. Apply integration by parts on a partition of $\mathcal{J}$:
$$(v, \mathcal{J} e)_\Omega \overbrace{=}^{i.b.p.} j(v, e)_\Omega + b(v, u_\partial)_{\partial\Omega},$$
so that $j(v, e)_\Omega$ is a skew-symmetric bilinear form.
3. Discretization by Galerkin method (same basis function for test and co-energy variables)

Application to the wave equation

If the integration by parts is applied on $\mathcal{J}_{\text{div}}$

$$(w, \mathcal{J} e)_{\Omega_r} = (\boldsymbol{w}_v, \mathrm{grad}_r e_p)_{\Omega_r} - (\mathrm{grad}_r w_p, \boldsymbol{e}_v)_{\Omega_r} + (w_p, u_N)_{\partial\Omega_r} .$$

The skew-symmetric bilinear form

$$j_{\text{grad}}(w, e) := (\boldsymbol{w}_v, \mathrm{grad}_r e_p)_{\Omega_r} - (\mathrm{grad}_r w_p, \boldsymbol{e}_v)_{\Omega_r}$$

is introduced, together with the boundary form

$$(w_p, u_N)_{\partial\Omega_r} = \int_{\partial\Omega_r} w_p u_N \; \mathrm{d}\Gamma_r,$$

where $u_N = \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega_r}$. The corresponding power conjugated output is given by $y_N = p|_{\partial\Omega_r}$. The system in weak form under Neumann boundary control is then written as

$$(w, \mathcal{M}\partial_t e)_{\Omega_r} = j_{\text{grad}}(w, e) + (w_p, u_N)_{\partial\Omega_r} .$$
$$(w_N, y_N)_{\partial\Omega_r} = (w_N, p)_{\partial\Omega_r} ,$$

Application to the wave equation

If the integration by parts is carried out on $\mathcal{J}_{\mathsf{grad}}$

$$(w, \mathcal{J}e)_{\Omega_r} = (w_p, \mathrm{div}_r\, \boldsymbol{e}_v)_{\Omega_r} - (\mathrm{div}_r\, \boldsymbol{w}_v, e_p)_{\Omega_r} + (\boldsymbol{w}_v \cdot \boldsymbol{n}, u_D)_{\partial\Omega_r}\,.$$

The skew-symmetric bilinear form

$$j_{\mathsf{div}}(w, e) := (w_p, \mathrm{div}_r\, \boldsymbol{e}_v)_{\Omega_r} - (\mathrm{div}_r\, \boldsymbol{w}_v, e_p)_{\Omega_r}$$

is introduced, together with the boundary form

$$(\boldsymbol{w}_v \cdot \boldsymbol{n}, u_D)_{\partial\Omega_r} = \int_{\partial\Omega_r} \boldsymbol{w}_v \cdot \boldsymbol{n}\; u_D \;\; \mathrm{d}\Gamma_r,$$

where $u_D = p\,|_{\partial\Omega_r}$. Adding the conjugated output $y_D = \boldsymbol{v} \cdot \; \boldsymbol{n}|_{\partial\Omega_r}$, the system in weak form under Dirichlet boundary control is then written as

$$\begin{aligned}(w, \mathcal{M}\partial_t e)_{\Omega_r} &= j_{\mathsf{div}}(w, e) + (\boldsymbol{w}_v \cdot \boldsymbol{n}, u_D)_{\partial\Omega_r}\,, \\ (w_D, y_D)_{\partial\Omega_r} &= (w_D, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial\Omega_r}\,,\end{aligned}$$

Mixed boundary condition

To tackle mixed boundary conditions two approaches are developed:

- a Lagrange multiplier based method;
- a virtual domain decomposition method.

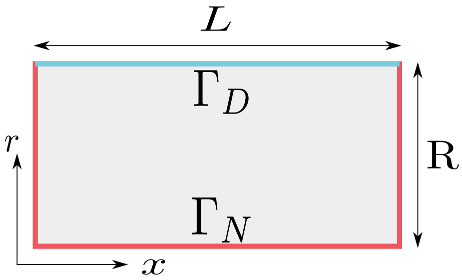


Figure: Boundary partition for the problem.

Plan

1 Introduction: problem statement

2 Finite dimensional discretization

- Lagrange multiplier approach
- Virtual domain decomposition

3 Results

$$(w, \mathcal{J}e)_{\Omega_r} = j_{\text{grad}}(w, e) + (w_p, \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega_r})_{\partial\Omega_r} .$$

The quantity $\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega_r}$ is known on Γ_N only. On Γ_D the Lagrange multiplier λ_D is introduced

$$\int_{\partial\Omega_r} w_p \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\Gamma_r = \int_{\Gamma_N} w_p u_N \, \mathrm{d}\Gamma_r + \int_{\Gamma_D} w_p \lambda_D \, \mathrm{d}\Gamma_r .$$

The constraint is the non-homogeneous Dirichlet condition

$$\int_{\Gamma_D} w_\lambda (p - u_D) \, \mathrm{d}\Gamma_r = 0,$$ w_λ test function for the Lagrange multiplier.

Final Weak Form with Lagrange multiplier

$$\begin{aligned}
m(w, \partial_t e) &= j_{\text{grad}}(w, e) + (w_p, \lambda_D)_{\Gamma_D} + (w_p, u_N)_{\Gamma_N} , \\
0 &= -(w_\lambda, p)_{\Gamma_D} + (w_\lambda, u_D)_{\Gamma_D} , \\
(w_N, y_N)_{\Gamma_N} &= (w_N, p)_{\Gamma_N} , \\
(w_D, y_D)_{\Gamma_D} &= (w_D, \lambda_D)_{\Gamma_D} ,
\end{aligned}$$

Lagrange multiplier approach

A Galerkin method can now be applied to retrieve a finite dimensional pH system. This means that corresponding test and trial functions are discretized using the same basis

$$
\begin{aligned}
p &\approx \sum_{i=1}^{n_p} \phi_p^i(x,r)p^i, \quad *_D \approx \sum_{i=1}^{n_D} \phi_\Gamma^i(s_D)*_D^i, \quad s_D \in \Gamma_D, \quad (* = \{u, y, \lambda\}), \\
\boldsymbol{v} &\approx \sum_{i=1}^{n_v} \boldsymbol{\phi}_v^i(x,r)v^i, \quad *_N \approx \sum_{i=1}^{n_N} \phi_\Gamma^i(s_N)*_N^i, \quad s_N \in \Gamma_N, \quad (* = \{u, y\}).
\end{aligned}
$$

A pHDAE system is obtained:

$$
\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{e} \\ \boldsymbol{\lambda}_D \end{pmatrix} = \begin{bmatrix} \mathbf{J} & \mathbf{G}_D \\ -\mathbf{G}_D^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e} \\ \boldsymbol{\lambda}_D \end{pmatrix} + \begin{bmatrix} \mathbf{B}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_D \end{bmatrix} \begin{pmatrix} \mathbf{u}_N \\ \mathbf{u}_D \end{pmatrix},
$$

$$
\begin{bmatrix} \mathbf{M}_{\Gamma_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\Gamma_D} \end{bmatrix} \begin{pmatrix} \mathbf{y}_N \\ \mathbf{y}_D \end{pmatrix} = \begin{bmatrix} \mathbf{B}_N^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_D^\top \end{bmatrix} \begin{pmatrix} \mathbf{e} \\ \boldsymbol{\lambda}_D \end{pmatrix}.
$$

Imposition of the boundary conditions

Take the weak form of $u_D = -\mathcal{Z}\lambda_D = -\mathcal{Z}y_D$:

$$\mathbf{M}_{\Gamma_D}\mathbf{u}_D = -\mathbf{M}_{\Gamma_D,\mathcal{Z}}\mathbf{y}_D,$$

This amounts to applying the control law

$$\mathbf{u}_D = -\mathbf{Z}\mathbf{B}_D^T\boldsymbol{\lambda}_D, \qquad \mathbf{Z} = \mathbf{M}_{\Gamma_D}^{-1}\mathbf{M}_{\Gamma_D,\mathcal{Z}}\mathbf{M}_{\Gamma_D}^{-1}$$

The Neumann boundary condition is imposed projecting $u_N = f(r)$.

Finite dimensional system with Lagrange multiplier

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{e} \\ \boldsymbol{\lambda}_D \end{bmatrix} = \begin{bmatrix} \mathbf{J} & \mathbf{G}_D \\ -\mathbf{G}_D^\top & -\mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \boldsymbol{\lambda}_D \end{bmatrix} + \begin{bmatrix} \mathbf{b}_N \\ \mathbf{0} \end{bmatrix},$$

with $\mathbf{R} = \mathbf{B}_D\mathbf{Z}\mathbf{B}_D^T$ a symmetric positive definite matrix.

Plan

Decomposition of the domain

First of all the domain has to be decomposed.
The interface between the two subdomain is chosen to get regular meshes on both subdomains.

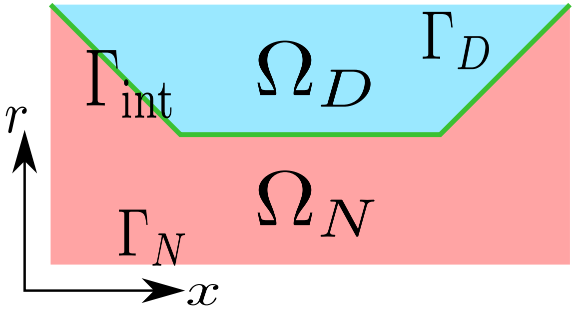


Figure: Virtual decomposition of the domain.

Virtual domain decomposition

Two weak formulations are constructed:

$$(w, \mathcal{M}\partial_t e)_{\Omega_N} = (w, \mathcal{J} e)_{\Omega_N}, \qquad \text{where } (\alpha, \beta)_{\Omega_N} = \int_{\Omega_N} \alpha \cdot \beta \, d\Omega_N,$$

$$(w, \mathcal{M}\partial_t e)_{\Omega_D} = (w, \mathcal{J} e)_{\Omega_D}, \qquad \text{where } (\alpha, \beta)_{\Omega_D} = \int_{\Omega_D} \alpha \cdot \beta \, d\Omega_D.$$

The integration by parts is performed differently on each subdomain to highlight the appropriate boundary input

$$(w, \mathcal{M} e)_{\Omega_N} = j_{\text{grad}}^{\Omega_N}(w, e) + (w_p, u_N)_{\partial\Omega_N},$$

$$(w, \mathcal{M} e)_{\Omega_D} = j_{\text{div}}^{\Omega_D}(w, e) + (\boldsymbol{w}_v \cdot \boldsymbol{n}, u_D)_{\partial\Omega_D},$$

where the bilinear skew-symmetric forms are defined on each subdomain

$$j_{\text{grad}}^{\Omega_N}(w, e) := (\boldsymbol{w}_v, \text{grad}_r e_p)_{\Omega_N} - (\text{grad}_r w_p, \boldsymbol{e}_v)_{\Omega_N},$$

$$j_{\text{div}}^{\Omega_D}(w, e) := (w_p, \text{div}_r \, \boldsymbol{e}_v)_{\Omega_D} - (\text{div}_r \, \boldsymbol{w}_v, e_p)_{\Omega_D}.$$

Virtual domain decomposition

The boundary terms are then split into two contributions

$$\partial\Omega_N = \Gamma_N \cup \Gamma_{\text{int}} \implies \quad (w_p, u_N)_{\partial\Omega_N} = (w_p, u_N)_{\Gamma_N} + (w_p, u_N)_{\Gamma_{\text{int}}},$$
$$\partial\Omega_D = \Gamma_D \cup \Gamma_{\text{int}} \implies (\boldsymbol{w}_v \cdot \boldsymbol{n}, u_D)_{\partial\Omega_D} = (\boldsymbol{w}_v \cdot \boldsymbol{n}, u_D)_{\Gamma_D} + (\boldsymbol{w}_v \cdot \boldsymbol{n}, u_D)_{\Gamma_{\text{int}}}.$$

Two finite dimensional pH systems are obtained

Subdomain Ω_N

$$\begin{aligned} \mathbf{M}_N \dot{\mathbf{e}}_N &= \mathbf{J}_N \mathbf{e}_N + \mathbf{B}_N \mathbf{u}_N + \mathbf{B}_N^{\text{int}} \mathbf{u}_N^{\text{int}}, \\ \mathbf{M}_{\Gamma_N} \mathbf{y}_N &= \mathbf{B}_N^\top \mathbf{e}_N, \\ \mathbf{M}_{\Gamma_{\text{int}}} \mathbf{y}_N^{\text{int}} &= \mathbf{B}_N^{\text{int}\,\top} \mathbf{e}_N, \end{aligned}$$

with Hamiltonian $H_{d,N} = \frac{1}{2}\mathbf{e}_N^\top \mathbf{M}_N \mathbf{e}_N$

Subdomain Ω_D

$$\begin{aligned} \mathbf{M}_D \dot{\mathbf{e}}_D &= \mathbf{J}_D \mathbf{e}_D + \mathbf{B}_D \mathbf{u}_D + \mathbf{B}_D^{\text{int}} \mathbf{u}_D^{\text{int}}, \\ \mathbf{M}_{\Gamma_D} \mathbf{y}_D &= \mathbf{B}_D^\top \mathbf{e}_D, \\ \mathbf{M}_{\Gamma_{\text{int}}} \mathbf{y}_D^{\text{int}} &= \mathbf{B}_D^{\text{int}\,\top} \mathbf{e}_D. \end{aligned}$$

with Hamiltonian $H_{d,D} = \frac{1}{2}\mathbf{e}_D^\top \mathbf{M}_D \mathbf{e}_D$

Power preserving interconnection

A gyrator interconnection is performed

$$\mathbf{u}_N^{\text{int}} = -\mathbf{y}_D^{\text{int}} = -\mathbf{M}_{\Gamma_{\text{int}}}^{-1} \mathbf{B}_D^{\text{int}\,\top} \mathbf{e}_D, \qquad \mathbf{u}_D^{\text{int}} = \mathbf{y}_N^{\text{int}} = \mathbf{M}_{\Gamma_{\text{int}}}^{-1} \mathbf{B}_N^{\text{int}\,\top} \mathbf{e}_N.$$

The interconnection implies that the power is exchanged without loss between the two systems

$$\mathbf{u}_D^{\text{int}\,\top} \mathbf{M}_{\Gamma_{\text{int}}} \mathbf{y}_D^{\text{int}} + \mathbf{u}_N^{\text{int}\,\top} \mathbf{M}_{\Gamma_{\text{int}}} \mathbf{y}_N^{\text{int}} = 0.$$

After imposition of the boundary condition the final system is obtained.

Finite dimensional system (Virtual domain decomposition)

$$\begin{bmatrix} \mathbf{M}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_D \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{e}_N \\ \mathbf{e}_D \end{bmatrix} = \left(\begin{bmatrix} \mathbf{J}_N & -\mathbf{C} \\ \mathbf{C}^\top & \mathbf{J}_D \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \right) \begin{bmatrix} \mathbf{e}_N \\ \mathbf{e}_D \end{bmatrix} + \begin{bmatrix} \mathbf{b}_N \\ \mathbf{0} \end{bmatrix}$$

with $\mathbf{C} = \mathbf{B}_N^{\text{int}} \mathbf{M}_{\Gamma_{\text{int}}}^{-1} \mathbf{B}_D^{\text{int}\,\top}$.

Plan

Physical interpretation of the impedance

The energy accounts for the pressure and velocity contribution

$$H_p = \frac{1}{2}\int \chi_s p^2 \, \mathrm{d}\Omega_r \approx \frac{1}{2}\mathbf{p}^T \mathbf{M}_p \mathbf{p}, \qquad H_v = \frac{1}{2}\int \mu_0 \, ||\mathbf{v}||^2 \, \mathrm{d}\Omega_r \approx \frac{1}{2}\mathbf{v}^T \mathbf{M}_v \mathbf{v},$$

The total energy at the initial time is the kinetic energy only

$$H_v^0 = H_{vx}^0 + H_{vr}^0 = \frac{1}{2}\int_0^L \int_0^R \mu_0 \left[(v_x^0)^2 + (v_r^0)^2\right] \, r \, \mathrm{d}r \, \mathrm{d}x.$$

The numerical values of the energy contribution are

$$H_v^0 = 0.453[J], \quad H_{vx}^0 = 0.204[J], \quad H_{vr}^0 = 0.249[J].$$

The impedance acts by dissipating the radial component of the velocity

$$\lim_{t\to\infty} H_{vr} \to 0, \qquad \lim_{t\to\infty} H_v \to H_{vx}^0 = 0.204[J]$$

Finite element choice

Pressure field approximation

The pressure $\phi_p(x, r)$ is interpolated using order 1 Lagrange polynomials.

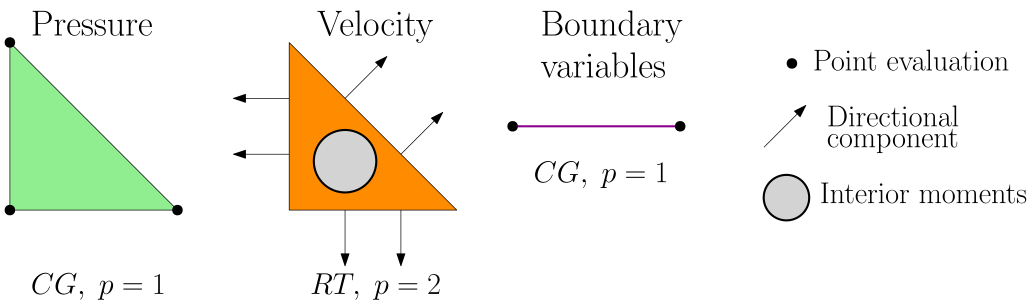
Velocity field approximation

The velocity field $\phi_v(x, r)$ is interpolated using order 2 Raviart-Thomas polynomials.

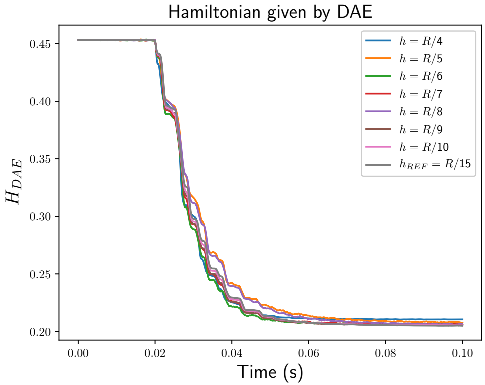
Boundary variables approximation

The boundary variables $\phi_\Gamma(s)$ are approximated by Lagrange polynomial of order 1 defined on the boundary Γ_D (for λ_D, u_D, y_D) or Γ_N (for u_N, y_N).

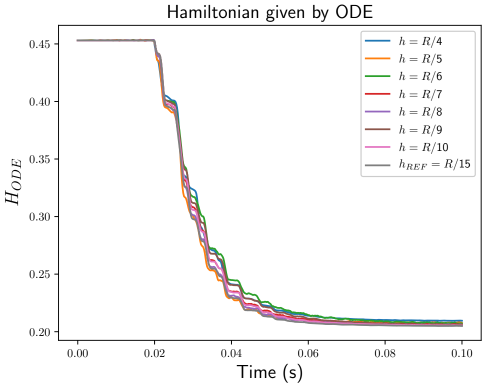
Finite element choice



Results



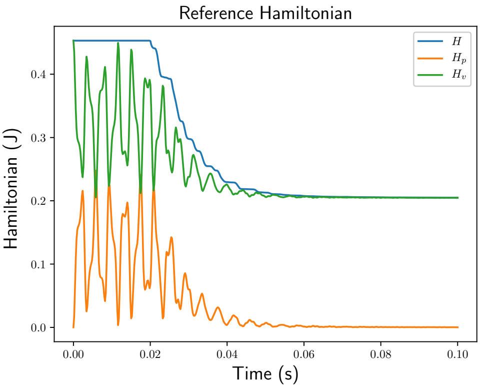
(a) DAE system.



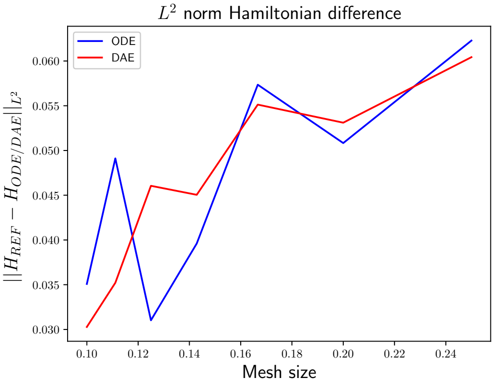
(b) ODE system.

Figure: Hamiltonian trend for different mesh size.

Results

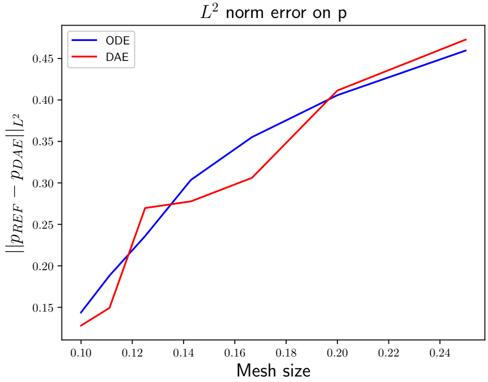


(a) Reference Hamiltonian.

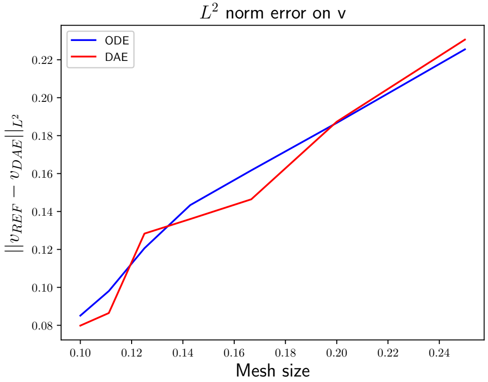


(b) L^2 Hamiltonian error.

Figure: Reference Hamiltonian and L^2 error.



(a) L^2 pressure error.



(b) L^2 velocity error.

Figure: Error on the state variables for different mesh size.

Conclusion

Future developments:

- a numerical analysis of the optimal choice for the underlying finite elements[1];
- the employment of theses techniques to more complicated models arising from structural and fluid mechanics;
- reformulation of the approach in terms of differential forms;
- application of the domain decomposition technique to parallelize simulations of large-scale models.

Thanks for your attention

[1] G. Haine, D. Matignon, and S. Anass. *A structure-preserving space-discretization for an anisotropic and heterogeneous boundary controlled N-dimensional wave equation as port-Hamiltonian system.* Submitted. 2020.

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Institut Supérieur de l'Aéronautique et de l'Espace

10 avenue Édouard Belin – BP 54032

31055 Toulouse Cedex 4 – France

Phone: +33 5 61 33 80 80

www.isae-supaero.fr