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Modelling and structure-preserving discretization of Maxwell's equations as port-Hamiltonian system

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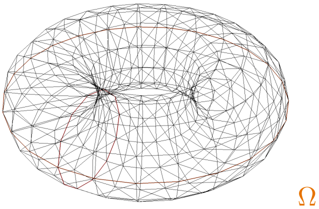
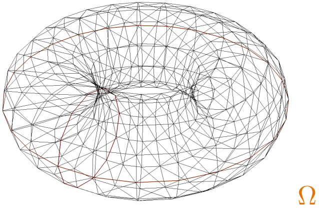
Outline

Outline

Modelling



- Electric permittivity: ϵ;
- Electric *induction*: $\mathbf{D}$, **energy variable**;
- Magnetic permeability: μ;
- Magnetic *induction*: $\mathbf{B}$, **energy variable**;
- Total inner distributed current: $\mathbf{J}$.



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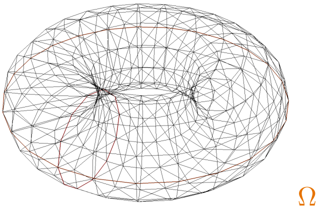
Hamiltonian = electromagnetic energy:

$$\mathcal{E}(\mathbf{D}, \mathbf{B}) := \frac{1}{2} \int_{\Omega} \frac{\mathbf{D} \cdot \mathbf{D}}{\epsilon} + \frac{\mathbf{B} \cdot \mathbf{B}}{\mu}.$$



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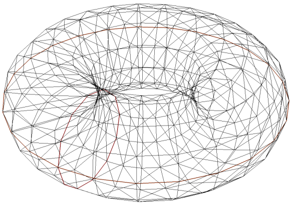
Co-energy variables, the electric and magnetic *fields*: $$\mathbf{E} := \delta_{\mathbf{D}}\mathcal{E} = \frac{\mathbf{D}}{\epsilon}, \qquad \mathbf{H} := \delta_{\mathbf{B}}\mathcal{E} = \frac{\mathbf{B}}{\mu}.$$



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Physical laws:

- Maxwell-Ampère: $\partial_t \mathbf{D} = \mathbf{curl}\ \mathbf{H} - \mathbf{J}$;
- Maxwell-Faraday: $\partial_t \mathbf{B} = -\mathbf{curl}\ \mathbf{E}$;
- Maxwell-Gauß (charge density): $\mathrm{div}\ \mathbf{D} = \rho$;
- Maxwell-flux: $\mathrm{div}\ \mathbf{B} = 0$;
- Ohm: $\mathbf{J} = \eta^{-1}\mathbf{E}$.

Outline

Power balance:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} &= \int_{\Omega} \left(\mathbf{E} \cdot \mathbf{curl}\ \mathbf{H} - \mathbf{H} \cdot \mathbf{curl}\ \mathbf{E}\right) - \int_{\Omega} \mathbf{E} \cdot \mathbf{J}, \\ &= -\int_{\partial\Omega} \mathrm{div}\left(\mathbf{E} \wedge \mathbf{H}\right) - \int_{\Omega} \mathbf{E} \cdot \mathbf{J}, \\ &= -\int_{\partial\Omega} \Pi \cdot \mathbf{n} - \int_{\Omega} \eta^{-1} \left\|\mathbf{E}\right\|^2.\end{aligned}$$

$\Pi := \gamma \left(\mathbf{E} \wedge \mathbf{H}\right)$ is the **Poynting** vector.

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The **variation** of energy is driven by:

- the flux of the Poynting vector across the boundary $\partial\Omega$;
- the loss in the thermal domain by Joule's effect distributed in the domain Ω.

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Collocated control $\boldsymbol{u}$ and observation $\boldsymbol{y}$ are taken on the boundary, such that $\boldsymbol{u}\cdot\boldsymbol{y} = -\Pi\cdot\mathbf{n}$.
Such a choice is called a *causality*.

Outline

Stokes-Dirac structure

Flows–efforts formulation: $\boldsymbol{f}_e := \partial_t \mathbf{D},\ \boldsymbol{f}_m := \partial_t \mathbf{B},\ \boldsymbol{f}_J := \mathbf{E}, \quad \boldsymbol{e}_e := \mathbf{E},\ \boldsymbol{e}_m := \mathbf{H},\ \boldsymbol{e}_J := \mathbf{J}.$

Electric control (voltage applied at the boundary): $\boldsymbol{e}_\partial = \boldsymbol{u} = (\mathbf{n} \wedge \mathbf{E}) \wedge \mathbf{n}.$

Magnetic observation (current at the boundary): $\boldsymbol{f}_\partial = -\boldsymbol{y} = \mathbf{H} \wedge \mathbf{n}.$

$$\boldsymbol{f}_\partial \cdot \boldsymbol{e}_\partial = \Pi \cdot \mathbf{n}.$$

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Structure operator:

$$\begin{pmatrix} \boldsymbol{f}_e \\ \boldsymbol{f}_m \\ \boldsymbol{f}_J \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \mathbf{curl} & -I \\ -\mathbf{curl} & 0 & 0 \\ I & 0 & 0 \end{bmatrix}}_{\mathcal{J}} \begin{pmatrix} \boldsymbol{e}_e \\ \boldsymbol{e}_m \\ \boldsymbol{e}_J \end{pmatrix},$$

Constitutive relations:

$$\boldsymbol{e}_e = \epsilon^{-1} \mathbf{D},$$
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Stokes-Dirac structure: *formal symmetry* $\mathbf{curl}^* = \mathbf{curl} \implies \mathcal{J}$ *formally skew-symmetric*:

$$\int_\Omega \boldsymbol{f}_e \cdot \boldsymbol{e}_e + \int_\Omega \boldsymbol{f}_m \cdot \boldsymbol{e}_m + \int_\Omega \boldsymbol{f}_J \cdot \boldsymbol{e}_J + \int_{\partial\Omega} \boldsymbol{f}_\partial \cdot \boldsymbol{e}_\partial = 0.$$

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Power balance: thanks to constitutive relations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E} = \int_\Omega \boldsymbol{f}_e \cdot \boldsymbol{e}_e + \int_\Omega \boldsymbol{f}_m \cdot \boldsymbol{e}_m = \underbrace{-\int_\Omega \eta \, \|\boldsymbol{f}_J\|^2}_{\text{Loss by Joule's effect}} \underbrace{-\int_{\partial\Omega} \boldsymbol{f}_\partial \cdot \boldsymbol{e}_\partial}_{\text{Control and observation}}.$$

Main results

- Application of the **Partitioned Finite Element Method** (PFEM) to mimic the Stokes-Dirac structure and the constitutive relations at the discrete level;
- Proof of the structure-preserving property: the **discrete power balance** reads as the continuous one;
- Efficient **implementations** with boundary control and internal damping (by Joule's effect) **with classical available open-source FEM softwares**, such as FreeFem++, FEniCS, XLiFE++, etc.

Outline

1 Introduction and continuous system

2 The Partitioned Finite Element Method (PFEM)
- Finite elements
- Final-dimensional Dirac structure and discrete power balance

3 Simulation results

4 Conclusion

The Partitioned Finite Element Method (PFEM)

The strategy follows:

1. Write the **weak formulation**;
2. Apply an appropriate **Stokes identity** (integration by parts) such that $\boldsymbol{u}$ “appears”;
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The weak formulation reads for all test functions $(\mathbf{\Phi^e}, \mathbf{\Phi^m}) \in \mathcal{H}_e \times \mathcal{H}_m$:

$$\begin{aligned} \int_\Omega \mathbf{\Phi^e} \cdot \partial_t \mathbf{D} &= \int_\Omega \mathbf{\Phi^e} \cdot \mathbf{curl}\, \mathbf{H} - \int_\Omega \mathbf{\Phi^e} \cdot \mathbf{J} \,, \\ \int_\Omega \mathbf{\Phi^m} \cdot \partial_t \mathbf{B} &= - \int_\Omega \mathbf{\Phi^m} \cdot \mathbf{curl}\, \mathbf{E} \,, \\ \int_\Omega \mathbf{\Phi^e} \cdot \boldsymbol{f}_J &= \int_\Omega \mathbf{\Phi^e} \cdot \mathbf{E} \,. \end{aligned}$$

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Intregating the *second* line by parts:

$$\int_\Omega \mathbf{\Phi^m} \cdot \partial_t \mathbf{B} = - \int_\Omega \mathbf{curl}\, \mathbf{\Phi^m} \cdot \mathbf{E} - \int_{\partial\Omega} \underbrace{(\mathbf{\Phi^m} \wedge \mathbf{n}) \cdot \mathbf{E}}_{=(\mathbf{\Phi^m}\wedge\mathbf{n})\cdot(\mathbf{n}\wedge\mathbf{E})\wedge\mathbf{n}=(\mathbf{\Phi^m}\wedge\mathbf{n})\cdot\boldsymbol{u}} \,.$$

The energy, co-energy, boundary and test functions of the *same* nature (electric, magnetic, or control and observation) are discretized by using the *same* vector-valued bases:

$$\mathbf{D}^d(\mathbf{x},t) := \sum_{i=1}^{N_e} D_i(t)\boldsymbol{\Phi}_i^{\mathbf{e}}(\mathbf{x}) = \boldsymbol{\Phi}^{\mathbf{e}\top}(\mathbf{x})\ \underline{D}(t)\ , \qquad \mathbf{E}^d(\mathbf{x},t) := \sum_{i=1}^{N_e} E_i(t)\boldsymbol{\Phi}_i^{\mathbf{e}}(\mathbf{x}) = \boldsymbol{\Phi}^{\mathbf{e}\top}(\mathbf{x})\ \underline{E}(t)\ ,$$

$$\boldsymbol{f}_J^d(\mathbf{x},t) := \sum_{i=1}^{N_e} f_i(t)\boldsymbol{\Phi}_i^{\mathbf{e}}(\mathbf{x}) = \boldsymbol{\Phi}^{\mathbf{e}\top}(\mathbf{x})\ \underline{f}(t)\ , \qquad \mathbf{J}^d(\mathbf{x},t) := \sum_{i=1}^{N_e} J_i(t)\boldsymbol{\Phi}_i^{\mathbf{e}}(\mathbf{x}) = \boldsymbol{\Phi}^{\mathbf{e}\top}(\mathbf{x})\ \underline{J}(t)\ ,$$

$$\mathbf{B}^d(\mathbf{x},t) := \sum_{k=1}^{N_m} B_k(t)\boldsymbol{\Phi}_k^{\mathbf{m}}(\mathbf{x}) = \boldsymbol{\Phi}^{\mathbf{m}\top}(\mathbf{x})\ \underline{B}(t)\ , \qquad \mathbf{H}^d(\mathbf{x},t) := \sum_{k=1}^{N_m} H_k(t)\boldsymbol{\Phi}_k^{\mathbf{m}}(\mathbf{x}) = \boldsymbol{\Phi}^{\mathbf{m}\top}(\mathbf{x})\ \underline{H}(t)\ ,$$

$$\boldsymbol{u}^d(\mathbf{s},t) := \sum_{m=1}^{N_\partial} u_m(t)\boldsymbol{\Psi}_m^{\partial}(\mathbf{s}) = \boldsymbol{\Psi}^{\partial\top}(\mathbf{s})\ \underline{u}(t)\ , \qquad \boldsymbol{y}^d(\mathbf{s},t) := \sum_{m=1}^{N_\partial} y_m(t)\boldsymbol{\Psi}_m^{\partial}(\mathbf{s}) = \boldsymbol{\Psi}^{\partial\top}(\mathbf{s})\ \underline{y}(t)\ .$$

with $\boldsymbol{\Phi}^{\mathbf{e}}$ an $N_e \times 3$ matrix, $\boldsymbol{\Phi}^{\mathbf{m}}$ an $N_m \times 3$ matrix and $\boldsymbol{\Psi}^{\partial}$ an $N_\partial \times 3$ matrix.

By injecting these discretizations in the weak formulation:

Discrete structure operator:

$$\begin{pmatrix} M_e \frac{\mathrm{d}}{\mathrm{d}t}\underline{D} \\ M_m \frac{\mathrm{d}}{\mathrm{d}t}\underline{B} \\ M_e \underline{f} \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & C & -M_e \\ -C^\top & 0 & 0 \\ M_e & 0 & 0 \end{bmatrix}}_{\mathcal{J}^d} \begin{pmatrix} \underline{E} \\ \underline{H} \\ \underline{J} \end{pmatrix} - \begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix} \underline{u},$$

Discrete constitutive relations:

$$\begin{aligned} M_e \, \underline{E} &= \langle \epsilon^{-1} \rangle \, \underline{D}, \\ M_m \, \underline{H} &= \langle \mu^{-1} \rangle \, \underline{B}, \\ M_e \, \underline{J} &= \langle \eta^{-1} \rangle \, \underline{f}. \end{aligned}$$

and the collocated observation is given by: $M_\partial \, \underline{y} = -\begin{bmatrix} 0 & T^\top & 0 \end{bmatrix} \begin{pmatrix} \underline{E}, & \underline{H}, & \underline{J} \end{pmatrix}^\top$, where:

$$(M_e)_{i,j} = \langle \mathbf{\Phi}_i^{\mathbf{e}}, \mathbf{\Phi}_j^{\mathbf{e}} \rangle_{L^2(\Omega)} \ , \quad (M_m)_{k,\ell} = \langle \mathbf{\Phi}_k^{\mathbf{m}}, \mathbf{\Phi}_\ell^{\mathbf{m}} \rangle_{L^2(\Omega)} \ , \quad (M_\partial)_{m,n} = \langle \mathbf{\Psi}_m^{\partial}, \mathbf{\Psi}_n^{\partial} \rangle_{L^2(\partial\Omega)} \ ,$$

$(C)_{i,\ell} = \langle \mathbf{\Phi}_i^{\mathbf{e}}, \mathbf{curl} \ \mathbf{\Phi}_\ell^{\mathbf{m}} \rangle_{L^2(\Omega)}$ of size $N_e \times N_m$, $\quad (T)_{k,n} = \langle (\mathbf{\Phi}_k^{\mathbf{m}} \wedge \mathbf{n}), \mathbf{\Psi}_n^{\partial} \rangle_{L^2(\partial\Omega)}$ of size $N_m \times N_\partial$,

and for the constitutive relations:

$$(\langle \epsilon^{-1} \rangle)_{i,j} = \langle \mathbf{\Phi}_i^{\mathbf{e}}, \epsilon^{-1} \mathbf{\Phi}_j^{\mathbf{e}} \rangle_{L^2(\Omega)} \ , \ (\langle \mu^{-1} \rangle)_{k,\ell} = \langle \mathbf{\Phi}_k^{\mathbf{m}}, \mu^{-1} \mathbf{\Phi}_\ell^{\mathbf{m}} \rangle_{L^2(\Omega)} \ , \ (\langle \eta^{-1} \rangle)_{i,j} = \langle \mathbf{\Phi}_i^{\mathbf{e}}, \eta^{-1} \mathbf{\Phi}_j^{\mathbf{e}} \rangle_{L^2(\Omega)} \ .$$

Outline

Dirac structure: by skew-symmetry of $\mathcal{J}^d$,

$$\underline{E}^\top M_e \frac{\mathrm{d}}{\mathrm{d}t}\underline{D} + \underline{H}^\top M_m \frac{\mathrm{d}}{\mathrm{d}t}\underline{B} + \underline{J}^\top M_e \underline{f} - \underline{u}^\top M_\partial \underline{y} = 0.$$

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Discrete Hamiltonian:

$$\mathcal{E}^d(\underline{D}, \underline{B}) := \mathcal{E}(\mathbf{D}^d, \mathbf{B}^d) = \frac{1}{2} \int_\Omega \frac{\mathbf{D}^d \cdot \mathbf{D}^d}{\epsilon} + \frac{\mathbf{B}^d \cdot \mathbf{B}^d}{\mu} = \frac{1}{2} \left(\underline{D}^\top \, \langle \epsilon^{-1} \rangle \, \underline{D} \, + \, \underline{B}^\top \, \langle \mu^{-1} \rangle \, \underline{B} \right).$$

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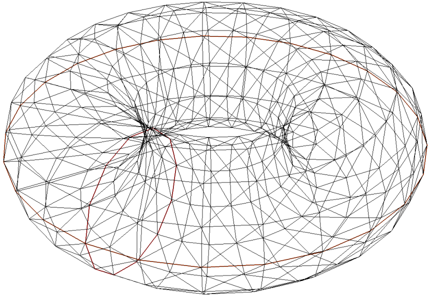
Discrete power balance:

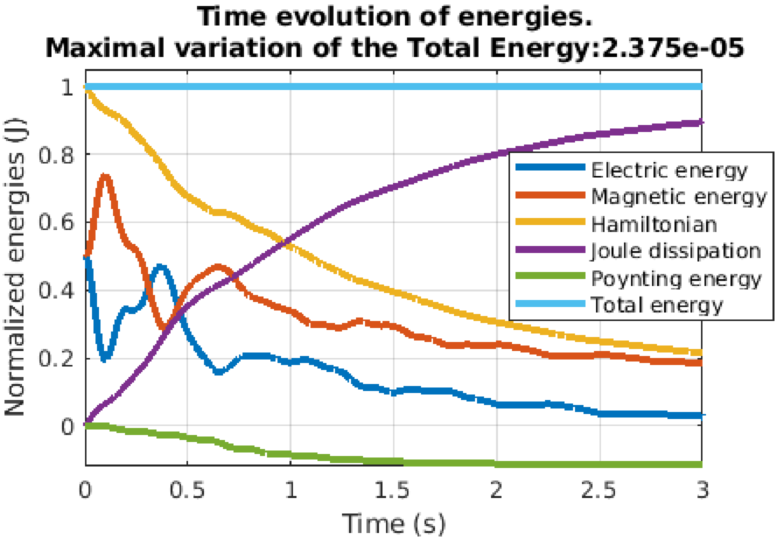
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^d(\underline{D}, \underline{B}) &= \underline{D}^\top \, \langle \epsilon^{-1} \rangle \, \frac{\mathrm{d}}{\mathrm{d}t}\underline{D} \, + \, \underline{B}^\top \, \langle \mu^{-1} \rangle \, \frac{\mathrm{d}}{\mathrm{d}t}\underline{B}, \\
&= \underline{E}^\top \, M_e \, \frac{\mathrm{d}}{\mathrm{d}t}\underline{D} \, + \, \underline{H}^\top \, M_m \, \frac{\mathrm{d}}{\mathrm{d}t}\underline{B}, \\
&= -\underline{E}^\top \, \langle \eta^{-1} \rangle \, \underline{E}^\top \, + \, \underline{u}^\top \, M_\partial \, \underline{y}.
\end{aligned}$$

Outline

- **Software:** FreeFem++ (v 4.4);
- **Finite elements $\mathbf{\Phi^e}$ and $\mathbf{\Phi^m}$:**
  first order Nédélec finite elements ($\mathbf{curl}$-conforming),
  $3 \times 15{,}144$ dof;
- **Boundary finite elements $\mathbf{\Psi}^{\partial}$:**
  discontinuous $\mathbb{P}^1$ Lagrange finite elements,
  11,346 dof;
- **Physical parameters:** $\epsilon = \mu = \eta = 1$.
- **Initial data:** divergence-free;
- **Boundary control:**
  time- and space-varying,
  compatible with the initial data;
- **Time scheme:**
  Crank-Nicolson with time step $\Delta t = 10^{-3}$.

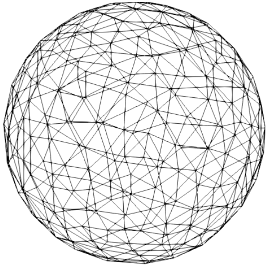


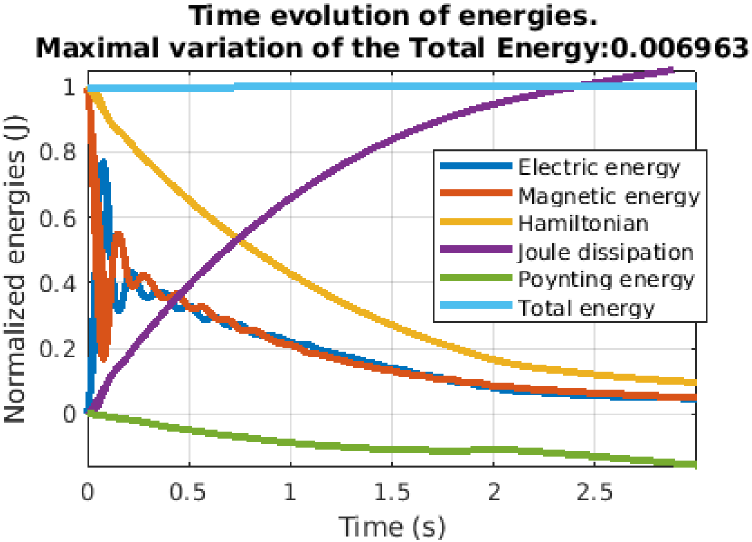
Simulation results



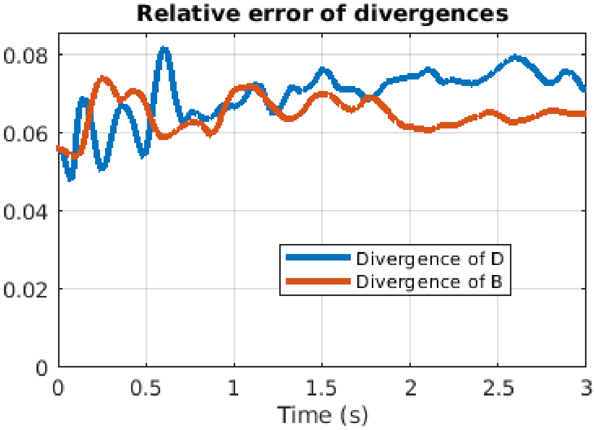
An other test, changing:

- **The domain:** a sphere;
- **The software:** FEniCS;
- **The time step:** $\Delta t = 10^{-2}$.





What about the divergences of the electric and magnetic inductions?



Approximations $\mathbf{D}^d$ and $\mathbf{B}^d$ are projected on first order Raviart-Thomas (div-conforming) finite elements.
Their divergences are compare to 0 *via*:

$$Error^2 = \frac{\|\mathrm{div}\ I\|^2_{L^2}}{\|I\|^2_{L^2} + \|\mathrm{div}\ I\|^2_{L^2}},$$

where $I = \mathbf{D}$ or $\mathbf{B}$.

Although non-exploding, the divergence-free property of inductions is not preserved by PFEM in its present form.

Outline

Conclusion

We propose a new approach for the structure-preserving discretization of Maxwell's equations.

- The **Partitioned Finite Element Method** (PFEM) has been extended to this problem;
- The **structure-preserving** property has been fully proved;
- **Simulations** in two different cases have been provided.

Further works:

- Structure-preserving discretization of the divergences (Differential Algebraic Equations);
- Control by charge density in Maxwell-Gauß's equation.

Thank you for your attention!

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