Geometric Science of Information
27th–29th August 2019, Toulouse

A Partioned Finite Element Method for the Structure-Preserving Discretization of Damped Infinite-Dimensional Port-Hamiltonian Systems with Boundary Control

Anass Serhani[1] **Denis Matignon**[1] **Ghislain Haine**[1]

[1]ISAE-Supaero, Toulouse

Outline

Outline

Main Objective

Aim:

Simulate complex open physical systems by ensuring the *conservation of the* ***power balance*** for a chosen functional: the **Hamiltonian**.

Main Objective

Aim:

Simulate complex open physical systems by ensuring the *conservation of the **power balance*** for a chosen functional: the **Hamiltonian**.

- *Finite Element Method:*
  $\rightarrow$ **Complex geometries** are allowed.
  $\rightarrow$ A wide range of **implementation** tools are available.

Main Objective

Aim:

Simulate complex open physical systems by ensuring the *conservation of the* ***power balance*** for a chosen functional: the **Hamiltonian**.

- *Finite Element Method:*
  $\rightarrow$ **Complex geometries** are allowed.
  $\rightarrow$ A wide range of **implementation** tools are available.

- *Port-Hamiltonian Systems (PHS):*
  $\rightarrow$ Model **"energy" exchanges** between simpler open subsystems.
  $\rightarrow$ The power balance is *encoded* in a **Stokes-Dirac structure**.

Main Objective

Aim:

Simulate complex open physical systems by ensuring the *conservation of the* ***power balance*** for a chosen functional: the **Hamiltonian**.

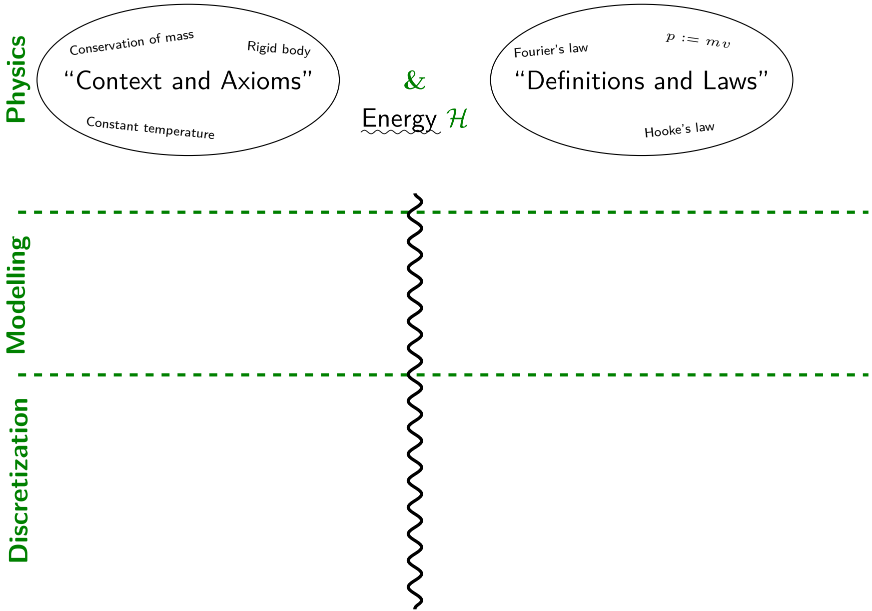
- *Finite Element Method:*
  $\rightarrow$ **Complex geometries** are allowed.
  $\rightarrow$ A wide range of **implementation** tools are available.

- *Port-Hamiltonian Systems (PHS):*
  $\rightarrow$ Model **"energy" exchanges** between simpler open subsystems.
  $\rightarrow$ The power balance is *encoded* in a **Stokes-Dirac structure**.

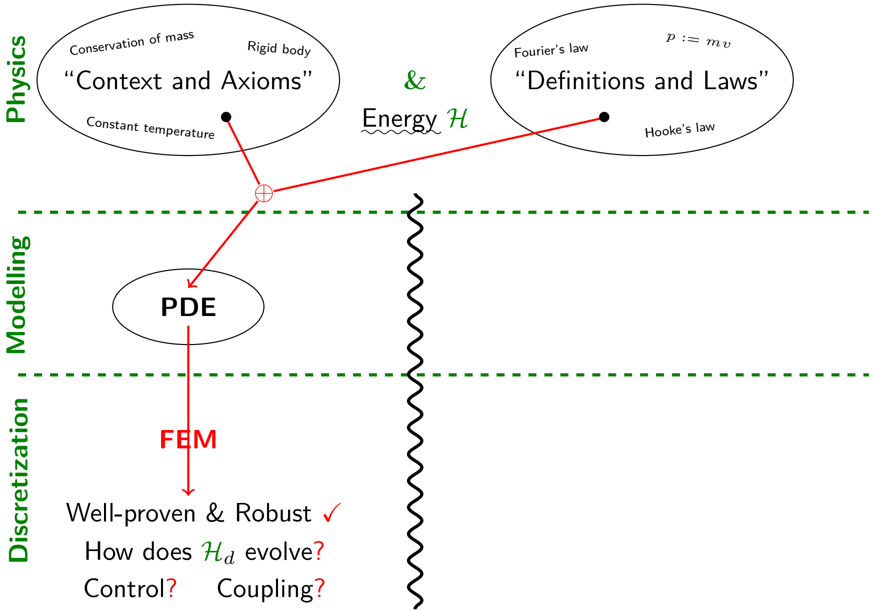
- *Partitioned Finite Element Method (PFEM):*
  $\rightarrow$ It translates the Stokes-Dirac structure into a **Dirac structure**.
  $\rightarrow$ The **discrete Hamiltonian** satisfies the "discrete" power balance.

A structure-preserving Partitioned Finite Element Method for the 2D wave equation
Cardoso-Ribeiro F.L., Matignon D., Lefèvre L.
IFAC-PapersOnLine, vol.51(3), pp.119–124 (2018), LHMNC 2018

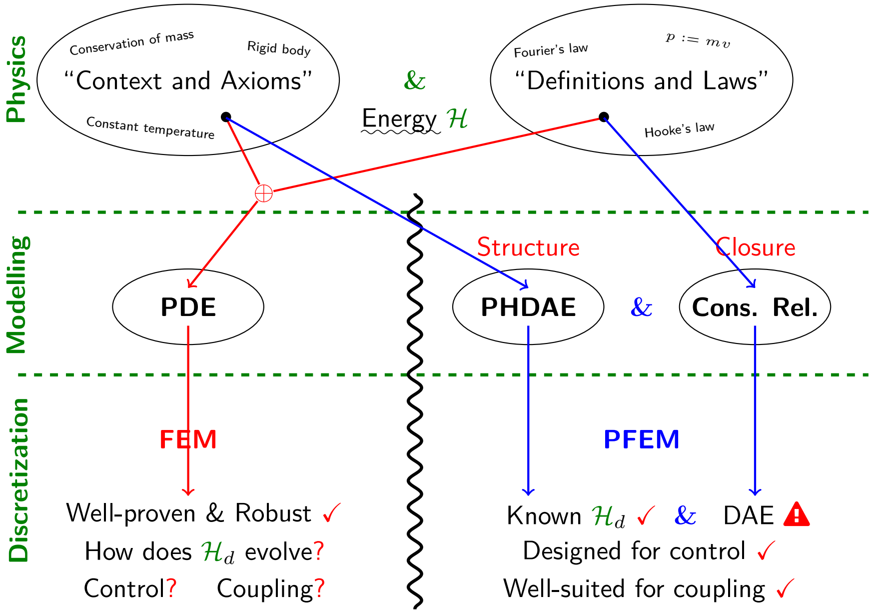
Change of paradigm?



Change of paradigm?



Change of paradigm?



Outline

Port-Hamiltonian Systems (PHS)

- The **energy variables** $\overrightarrow{\alpha}$ (vector field);
- The **Hamiltonian** $\mathcal{H}(\overrightarrow{\alpha}(t))$ (positive functional);

Port-Hamiltonian Systems (PHS)

- The **energy variables** $\overrightarrow{\alpha}$ (vector field);
- The **Hamiltonian** $\mathcal{H}(\overrightarrow{\alpha}(t))$ (positive functional);
- The **co-energy variables** $\overrightarrow{e}_{\overrightarrow{\alpha}} := \delta_{\overrightarrow{\alpha}}\mathcal{H}$ (vector field),
  $\rightsquigarrow$ the variational derivative of $\mathcal{H}$ w.r.t $\overrightarrow{\alpha}$;

Port-Hamiltonian Systems (PHS)

- The **energy variables** $\overrightarrow{\alpha}$ (vector field);
- The **Hamiltonian** $\mathcal{H}(\overrightarrow{\alpha}(t))$ (positive functional);
- The **co-energy variables** $\overrightarrow{e}_{\overrightarrow{\alpha}} := \delta_{\overrightarrow{\alpha}}\mathcal{H}$ (vector field),
  $\rightsquigarrow$ the variational derivative of $\mathcal{H}$ w.r.t $\overrightarrow{\alpha}$;
- The ***structure*** **operator** J (linear and *formally skew-symmetric*);
- The **resistive/dissipative operator** R (linear and *positive*);

Port-Hamiltonian Systems (PHS)

- The **energy variables** $\overrightarrow{\alpha}$ (vector field);
- The **Hamiltonian** $\mathcal{H}(\overrightarrow{\alpha}(t))$ (positive functional);
- The **co-energy variables** $\overrightarrow{e}_{\overrightarrow{\alpha}} := \delta_{\overrightarrow{\alpha}}\mathcal{H}$ (vector field),
  $\rightsquigarrow$ the variational derivative of $\mathcal{H}$ w.r.t $\overrightarrow{\alpha}$;
- The ***structure*** **operator** J (linear and *formally skew-symmetric*);
- The **resistive/dissipative operator** R (linear and *positive*);
- The **control operator** B (linear);
- The **input** $\boldsymbol{u}$ and the **collocated output** $\boldsymbol{y}$ (boundary scalar fields);

Port-Hamiltonian Systems (PHS)

- The **energy variables** $\overrightarrow{\alpha}$ (vector field);
- The **Hamiltonian** $\mathcal{H}(\overrightarrow{\alpha}(t))$ (positive functional);
- The **co-energy variables** $\overrightarrow{e}_{\overrightarrow{\alpha}} := \delta_{\overrightarrow{\alpha}}\mathcal{H}$ (vector field),
  $\rightsquigarrow$ the variational derivative of $\mathcal{H}$ w.r.t $\overrightarrow{\alpha}$;
- The ***structure* operator** J (linear and *formally skew-symmetric*);
- The **resistive/dissipative operator** R (linear and *positive*);
- The **control operator** B (linear);
- The **input** $\boldsymbol{u}$ and the **collocated output** $\boldsymbol{y}$ (boundary scalar fields);
- The **dynamical system**:

$$\begin{cases} \partial_t \overrightarrow{\alpha}(t) = (J - R)\,\overrightarrow{e}_{\overrightarrow{\alpha}}(t) + B\boldsymbol{u}(t), \\ \boldsymbol{y}(t) = B^* \overrightarrow{e}_{\overrightarrow{\alpha}}(t). \end{cases}$$

Port-Hamiltonian Systems (PHS)

- The **energy variables** $\vec{\alpha}$ (vector field);
- The **Hamiltonian** $\mathcal{H}(\vec{\alpha}(t))$ (positive functional);
- The **co-energy variables** $\vec{e}_{\vec{\alpha}} := \delta_{\vec{\alpha}} \mathcal{H}$ (vector field),
  $\rightsquigarrow$ the variational derivative of $\mathcal{H}$ w.r.t $\vec{\alpha}$;
- The ***structure* operator** J (linear and *formally skew-symmetric*);
- The **resistive/dissipative operator** R (linear and *positive*);
- The **control operator** B (linear);
- The **input** $\boldsymbol{u}$ and the **collocated output** $\boldsymbol{y}$ (boundary scalar fields);
- The **dynamical system**:

$$\left\{ \begin{array}{l} \partial_t \vec{\alpha}(t) = (J - R)\, \vec{e}_{\vec{\alpha}}(t) + B\boldsymbol{u}(t), \\ \boldsymbol{y}(t) = B^* \vec{e}_{\vec{\alpha}}(t). \end{array} \right.$$

Lossy Power Balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\vec{\alpha}(t)) = -\left\langle R\vec{e}_{\vec{\alpha}}(t), \vec{e}_{\vec{\alpha}}(t)\right\rangle_J + \langle \boldsymbol{u}(t), \boldsymbol{y}(t)\rangle_B \leq \langle \boldsymbol{u}(t), \boldsymbol{y}(t)\rangle_B \,.$$

⚠ Although **the underlying geometry** is well-determined with the above equality, **constitutive relations** between $\vec{\alpha}$ and $\vec{e}_{\vec{\alpha}}$ are also needed to solve the system!

Associated (Stokes-)Dirac structures

- The **effort space** $\mathcal{E}$ (Hilbert space) and $\overrightarrow{e} := \left(\overrightarrow{e}_{\overrightarrow{\alpha}}, \quad \overrightarrow{e}_R, \quad u\right)^\top$;

Associated (Stokes-)Dirac structures

- The **effort space** $\mathcal{E}$ (Hilbert space) and $\overrightarrow{\boldsymbol{e}} := \left(\overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}, \quad \overrightarrow{\boldsymbol{e}}_R, \quad \boldsymbol{u}\right)^\top$;
- The **flow space** $\mathcal{F} := \mathcal{E}'$ and $\overrightarrow{\boldsymbol{f}} := \left(\partial_t \overrightarrow{\boldsymbol{\alpha}}, \quad \overrightarrow{\boldsymbol{f}}_R, \quad -\boldsymbol{y}\right)^\top$;

Associated (Stokes-)Dirac structures

- The **effort space** $\mathcal{E}$ (Hilbert space) and $\overrightarrow{e} := \left(\overrightarrow{e}_{\overrightarrow{\alpha}}, \quad \overrightarrow{e}_R, \quad \boldsymbol{u}\right)^\top$;
- The **flow space** $\mathcal{F} := \mathcal{E}'$ and $\overrightarrow{f} := \left(\partial_t \overrightarrow{\alpha}, \quad \overrightarrow{f}_R, \quad -\boldsymbol{y}\right)^\top$;
- The ***input–output structure* operator** $\mathcal{J} := \begin{pmatrix} J & -I & B \\ I & 0 & 0 \\ -B^* & 0 & 0 \end{pmatrix}$;

Associated (Stokes-)Dirac structures

- The **effort space** $\mathcal{E}$ (Hilbert space) and $\vec{e} := \left(\vec{e}_{\vec{\alpha}}, \quad \vec{e}_R, \quad \boldsymbol{u}\right)^\top$;
- The **flow space** $\mathcal{F} := \mathcal{E}'$ and $\vec{f} := \left(\partial_t \vec{\alpha}, \quad \vec{f}_R, \quad -\boldsymbol{y}\right)^\top$;
- The ***input–output structure* operator** $\mathcal{J} := \begin{pmatrix} J & -I & B \\ I & 0 & 0 \\ -B^* & 0 & 0 \end{pmatrix}$;
- The **Bond space** $\mathcal{B} := \mathcal{F} \times \mathcal{E}$, with symmetrized bilinear product:

$$\left[\begin{pmatrix} \vec{f}^{\,1} \\ \vec{e}^{\,1} \end{pmatrix}, \begin{pmatrix} \vec{f}^{\,2} \\ \vec{e}^{\,2} \end{pmatrix} \right]_{\mathcal{B}} := \left\langle \vec{f}^{\,1}, \vec{e}^{\,2} \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \vec{f}^{\,2}, \vec{e}^{\,1} \right\rangle_{\mathcal{F},\mathcal{E}};$$

Associated (Stokes-)Dirac structures

- The **effort space** $\mathcal{E}$ (Hilbert space) and $\vec{e} := (\vec{e}_{\vec{\alpha}}, \quad \vec{e}_R, \quad \boldsymbol{u})^\top$;
- The **flow space** $\mathcal{F} := \mathcal{E}'$ and $\vec{f} := \left(\partial_t \vec{\alpha}, \quad \vec{f}_R, \quad -\boldsymbol{y}\right)^\top$;
- The ***input–output structure* operator** $\mathcal{J} := \begin{pmatrix} J & -I & B \\ I & 0 & 0 \\ -B^* & 0 & 0 \end{pmatrix}$;
- The **Bond space** $\mathcal{B} := \mathcal{F} \times \mathcal{E}$, with symmetrized bilinear product:

$$\left[\begin{pmatrix} \vec{f}^1 \\ \vec{e}^1 \end{pmatrix}, \begin{pmatrix} \vec{f}^2 \\ \vec{e}^2 \end{pmatrix} \right]_{\mathcal{B}} := \left\langle \vec{f}^1, \vec{e}^2 \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \vec{f}^2, \vec{e}^1 \right\rangle_{\mathcal{F},\mathcal{E}};$$

- The **Dirac structure** $\mathcal{D} := \mathrm{Graph}\left(\mathcal{J}\right) \subset \mathcal{B}$, *i.e.* $\mathcal{D}^{[\perp]} = \mathcal{D}$ with:

$$\mathcal{D}^{[\perp]} := \left\{ \begin{pmatrix} \vec{f} \\ \vec{e} \end{pmatrix} \in \mathcal{B} \;\middle|\; \left[\begin{pmatrix} \vec{f} \\ \vec{e} \end{pmatrix}, \begin{pmatrix} \widetilde{\vec{f}} \\ \widetilde{\vec{e}} \end{pmatrix} \right]_{\mathcal{B}} = 0, \quad \forall \begin{pmatrix} \widetilde{\vec{f}} \\ \widetilde{\vec{e}} \end{pmatrix} \in \mathcal{D} \right\}.$$

⚠ Hypotheses on J and B are needed for $\mathcal{D}$ to be a (Stokes-)Dirac structure!

Associated (Stokes-)Dirac structures

- The **effort space** $\mathcal{E}$ (Hilbert space) and $\overrightarrow{e} := \left(\overrightarrow{e}_{\overrightarrow{\alpha}}, \quad \overrightarrow{e}_R, \quad \boldsymbol{u}\right)^\top$;
- The **flow space** $\mathcal{F} := \mathcal{E}'$ and $\overrightarrow{f} := \left(\partial_t \overrightarrow{\alpha}, \quad \overrightarrow{f}_R, \quad -\boldsymbol{y}\right)^\top$;
- The ***input–output structure* operator** $\mathcal{J} := \begin{pmatrix} J & -I & B \\ I & 0 & 0 \\ -B^* & 0 & 0 \end{pmatrix}$;
- The **Bond space** $\mathcal{B} := \mathcal{F} \times \mathcal{E}$, with symmetrized bilinear product:

$$\left[\begin{pmatrix} \overrightarrow{f}^1 \\ \overrightarrow{e}^1 \end{pmatrix}, \begin{pmatrix} \overrightarrow{f}^2 \\ \overrightarrow{e}^2 \end{pmatrix} \right]_{\mathcal{B}} := \left\langle \overrightarrow{f}^1, \overrightarrow{e}^2 \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \overrightarrow{f}^2, \overrightarrow{e}^1 \right\rangle_{\mathcal{F},\mathcal{E}};$$

- The **Dirac structure** $\mathcal{D} := \mathrm{Graph}\left(\mathcal{J}\right) \subset \mathcal{B}$, *i.e.* $\mathcal{D}^{[\perp]} = \mathcal{D}$ with:

$$\mathcal{D}^{[\perp]} := \left\{ \begin{pmatrix} \overrightarrow{f} \\ \overrightarrow{e} \end{pmatrix} \in \mathcal{B} \;\middle|\; \left[\begin{pmatrix} \overrightarrow{f} \\ \overrightarrow{e} \end{pmatrix}, \begin{pmatrix} \widetilde{\overrightarrow{f}} \\ \widetilde{\overrightarrow{e}} \end{pmatrix} \right]_{\mathcal{B}} = 0, \quad \forall \begin{pmatrix} \widetilde{\overrightarrow{f}} \\ \widetilde{\overrightarrow{e}} \end{pmatrix} \in \mathcal{D} \right\}.$$

- The **dissipative constitutive relation** $\overrightarrow{e}_R = R \overrightarrow{f}_R$;

⚠ Hypotheses on J and B are needed for $\mathcal{D}$ to be a (Stokes-)Dirac structure!

Resulting Differential Algebraic Equations (DAE)

$$\left\langle \overrightarrow{f}(t), \overrightarrow{e}(t) \right\rangle_{\mathcal{F},\mathcal{E}} = 0, \quad \forall \left(\overrightarrow{f}(t), \overrightarrow{e}(t) \right) \in \mathcal{D}, \quad \forall t \geq 0.$$

Resulting Differential Algebraic Equations (DAE)

$$\left\langle \overrightarrow{\boldsymbol{f}}(t), \overrightarrow{\boldsymbol{e}}(t) \right\rangle_{\mathcal{F},\mathcal{E}} = 0, \quad \forall \left(\overrightarrow{\boldsymbol{f}}(t), \overrightarrow{\boldsymbol{e}}(t) \right) \in \mathcal{D}, \quad \forall t \geq 0.$$

Let $\left(\partial_t \overrightarrow{\boldsymbol{\alpha}}, \overrightarrow{\boldsymbol{f}}_R, -\boldsymbol{y}, \overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}, \overrightarrow{\boldsymbol{e}}_R, \boldsymbol{u} \right)^\top$ be in $\mathcal{D}$.

Adding $\overrightarrow{\boldsymbol{e}}_R = R \overrightarrow{\boldsymbol{f}}_R$: **the lossy power balance is satisfied**!

$$\left\langle \overrightarrow{\boldsymbol{f}}(t), \overrightarrow{\boldsymbol{e}}(t) \right\rangle_{\mathcal{F},\mathcal{E}} = 0, \quad \forall \left(\overrightarrow{\boldsymbol{f}}(t), \overrightarrow{\boldsymbol{e}}(t) \right) \in \mathcal{D}, \quad \forall t \geq 0.$$

Let $\left(\partial_t \overrightarrow{\boldsymbol{\alpha}}, \overrightarrow{\boldsymbol{f}}_R, -\boldsymbol{y}, \overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}, \overrightarrow{\boldsymbol{e}}_R, \boldsymbol{u} \right)^\top$ be in $\mathcal{D}$.

Adding $\overrightarrow{\boldsymbol{e}}_R = R \overrightarrow{\boldsymbol{f}}_R$: **the lossy power balance is satisfied**!

$\Longrightarrow$ Flow/effort representation generalizes the above state representation with $J - R$, *and PFEM appears to be very well-suited to it!*

PHS + DAE = **PHDAE**.

$$\left\langle \overrightarrow{\boldsymbol{f}}(t), \overrightarrow{\boldsymbol{e}}(t) \right\rangle_{\mathcal{F},\mathcal{E}} = 0, \quad \forall \left(\overrightarrow{\boldsymbol{f}}(t), \overrightarrow{\boldsymbol{e}}(t) \right) \in \mathcal{D}, \quad \forall t \geq 0.$$

Let $\left(\partial_t \overrightarrow{\boldsymbol{\alpha}}, \overrightarrow{\boldsymbol{f}}_R, -\boldsymbol{y}, \overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}, \overrightarrow{\boldsymbol{e}}_R, \boldsymbol{u} \right)^\top$ be in $\mathcal{D}$.

Adding $\overrightarrow{\boldsymbol{e}}_R = R \overrightarrow{\boldsymbol{f}}_R$: **the lossy power balance is satisfied**!

$\Longrightarrow$ Flow/effort representation generalizes the above state representation with $J - R$, *and PFEM appears to be very well-suited to it!*

PHS + DAE = **PHDAE**.

Main result

PFEM gives rise to a **finite-dimensional Dirac structure** *containing* a **discrete version of the (lossy) power balance** for the **discrete Hamiltonian**.

Outline

Conservative System: Wave as PHDAE

Deflection w **of a 2D-membrane, *boundary deflection velocity* as control.**

Conservative System: Wave as PHDAE

Deflection w of a 2D-membrane, *boundary deflection velocity* as control.

Its total energy is given by the sum of the potential & kinetic energies:

$$\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2} \int_\Omega \left(\overrightarrow{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q + \frac{1}{\rho} \alpha_p^2 \right).$$

Conservative System: Wave as PHDAE

Deflection w **of a 2D-membrane,** ***boundary deflection velocity*** **as control.**

Its total energy is given by the sum of the potential & kinetic energies:

$$\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2} \int_\Omega \left(\overrightarrow{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q + \frac{1}{\rho} \alpha_p^2 \right).$$

- ρ the *mass density* of the medium and $\overline{\overline{\boldsymbol{T}}}$ the *Young modulus* tensor;
- $\alpha_p := \rho \partial_t w$ the *linear momentum* and $\overrightarrow{\boldsymbol{\alpha}}_q := \overrightarrow{\mathbf{grad}}(w)$ the *strain*;

Deflection w of a 2D-membrane, *boundary deflection velocity* as control.

Its total energy is given by the sum of the potential & kinetic energies:

$$\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2}\int_\Omega \left(\overrightarrow{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q + \frac{1}{\rho}\alpha_p^2 \right).$$

- ρ the *mass density* of the medium and $\overline{\overline{\boldsymbol{T}}}$ the *Young modulus* tensor;
- $\alpha_p := \rho \partial_t w$ the *linear momentum* and $\overrightarrow{\boldsymbol{\alpha}}_q := \overrightarrow{\mathbf{grad}}\,(w)$ the *strain*;
- $\overrightarrow{\boldsymbol{e}}_q := \delta_{\overrightarrow{\boldsymbol{\alpha}}_q}\mathcal{H} = \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q$ the *stress*;
- $e_p := \delta_{\alpha_p}\mathcal{H} = \frac{\alpha_p}{\rho}$ the *deflection velocity* and $\boldsymbol{u} := e_p$;

Conservative System: Wave as PHDAE

Deflection w **of a 2D-membrane,** ***boundary deflection velocity*** **as control.**

Its total energy is given by the sum of the potential & kinetic energies:

$$\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2} \int_\Omega \left(\overrightarrow{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q + \frac{1}{\rho} \alpha_p^2 \right).$$

- ρ the *mass density* of the medium and $\overline{\overline{\boldsymbol{T}}}$ the *Young modulus* tensor;
- $\alpha_p := \rho \partial_t w$ the *linear momentum* and $\overrightarrow{\boldsymbol{\alpha}}_q := \overrightarrow{\mathbf{grad}}(w)$ the *strain*;
- $\overrightarrow{\boldsymbol{e}}_q := \delta_{\overrightarrow{\boldsymbol{\alpha}}_q} \mathcal{H} = \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q$ the *stress*;
- $e_p := \delta_{\alpha_p} \mathcal{H} = \frac{\alpha_p}{\rho}$ the *deflection velocity* and $\boldsymbol{u} := e_p$;
- $\boldsymbol{y} := \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}$ the output *normal stress*.

Conservative System: Wave as PHDAE

Deflection w **of a 2D-membrane,** ***boundary deflection velocity*** **as control.**

Its total energy is given by the sum of the potential & kinetic energies:

$$\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2} \int_\Omega \left(\overrightarrow{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q + \frac{1}{\rho} \alpha_p^2 \right).$$

- ρ the *mass density* of the medium and $\overline{\overline{\boldsymbol{T}}}$ the *Young modulus* tensor;
- $\alpha_p := \rho \partial_t w$ the *linear momentum* and $\overrightarrow{\boldsymbol{\alpha}}_q := \overrightarrow{\mathbf{grad}}(w)$ the *strain*;
- $\overrightarrow{\boldsymbol{e}}_q := \delta_{\overrightarrow{\boldsymbol{\alpha}}_q} \mathcal{H} = \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q$ the *stress*;
- $e_p := \delta_{\alpha_p} \mathcal{H} = \frac{\alpha_p}{\rho}$ the *deflection velocity* and $\boldsymbol{u} := e_p$;
- $\boldsymbol{y} := \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}$ the output *normal stress*.

$$\begin{cases} \rho \partial_{tt}^2 w &= \mathrm{div} \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}}(w) \right), \\ \boldsymbol{u} &= \partial_t w, \\ \boldsymbol{y} &= \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}}(w) \right) \cdot \overrightarrow{\boldsymbol{n}}. \end{cases}$$

Deflection w of a 2D-membrane, *boundary deflection velocity* as control.

Its total energy is given by the sum of the potential & kinetic energies:

$$\mathcal{H}(\vec{\alpha}_q, \alpha_p) := \frac{1}{2}\int_\Omega \left(\vec{\alpha}_q \cdot \overline{\overline{T}} \cdot \vec{\alpha}_q + \frac{1}{\rho}\alpha_p^2 \right).$$

- ρ the *mass density* of the medium and $\overline{\overline{T}}$ the *Young modulus* tensor;
- $\alpha_p := \rho \partial_t w$ the *linear momentum* and $\vec{\alpha}_q := \overrightarrow{\mathbf{grad}}(w)$ the *strain*;
- $\vec{e}_q := \delta_{\vec{\alpha}_q}\mathcal{H} = \overline{\overline{T}} \cdot \vec{\alpha}_q$ the *stress*;
- $e_p := \delta_{\alpha_p}\mathcal{H} = \frac{\alpha_p}{\rho}$ the *deflection velocity* and $\boldsymbol{u} := e_p$;
- $\boldsymbol{y} := \vec{e}_q \cdot \vec{n}$ the output *normal stress*.

$$\begin{cases} \rho \partial_{tt}^2 w &= \mathrm{div}\left(\overline{\overline{T}} \cdot \overrightarrow{\mathbf{grad}}(w)\right), \\ \boldsymbol{u} &= \partial_t w, \\ \boldsymbol{y} &= \left(\overline{\overline{T}} \cdot \overrightarrow{\mathbf{grad}}(w)\right) \cdot \vec{n}. \end{cases} \Leftrightarrow \begin{cases} \partial_t \vec{\alpha}_q &= \overrightarrow{\mathbf{grad}}(e_p), \\ \partial_t \alpha_p &= \mathrm{div}\left(\vec{e}_q\right), \end{cases} \& \begin{cases} \boldsymbol{u} &= e_p, \\ \boldsymbol{y} &= \vec{e}_q \cdot \vec{n}. \end{cases}$$

Deflection w of a 2D-membrane, *boundary deflection velocity* as control.

Its total energy is given by the sum of the potential & kinetic energies:

$$\mathcal{H}(\vec{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2} \int_\Omega \left(\vec{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{\alpha}}_q + \frac{1}{\rho} \alpha_p^2 \right).$$

- ρ the *mass density* of the medium and $\overline{\overline{\boldsymbol{T}}}$ the *Young modulus* tensor;
- $\alpha_p := \rho \partial_t w$ the *linear momentum* and $\vec{\boldsymbol{\alpha}}_q := \overrightarrow{\mathbf{grad}}(w)$ the *strain*;
- $\vec{\boldsymbol{e}}_q := \delta_{\vec{\boldsymbol{\alpha}}_q} \mathcal{H} = \overline{\overline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{\alpha}}_q$ the *stress*;
- $e_p := \delta_{\alpha_p} \mathcal{H} = \frac{\alpha_p}{\rho}$ the *deflection velocity* and $\boldsymbol{u} := e_p$;
- $\boldsymbol{y} := \vec{\boldsymbol{e}}_q \cdot \vec{\boldsymbol{n}}$ the output *normal stress*.

$$\left\{ \begin{aligned} \rho \partial_{tt}^2 w &= \operatorname{div}\left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}}(w) \right), \\ \boldsymbol{u} &= \partial_t w, \\ \boldsymbol{y} &= \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}}(w) \right) \cdot \vec{\boldsymbol{n}}. \end{aligned} \right. \Leftrightarrow \left\{ \begin{aligned} \partial_t \vec{\boldsymbol{\alpha}}_q &= \overrightarrow{\mathbf{grad}}(e_p), \\ \partial_t \alpha_p &= \operatorname{div}\left(\vec{\boldsymbol{e}}_q \right), \end{aligned} \right. \quad \& \left\{ \begin{aligned} \boldsymbol{u} &= e_p, \\ \boldsymbol{y} &= \vec{\boldsymbol{e}}_q \cdot \vec{\boldsymbol{n}}. \end{aligned} \right.$$

Lossless Power Balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}(\vec{\boldsymbol{\alpha}}_q, \alpha_p) = \langle \boldsymbol{y}, \boldsymbol{u} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}.$$

The strategy follows:

Conservative System: PFEM strategy

The strategy follows:

1. Write the **weak formulation**;

Conservative System: PFEM strategy

The strategy follows:

1. Write the **weak formulation**;
2. Apply an accurate **Stokes (Green) identity** (such that $\boldsymbol{u}$ "appears");

Conservative System: PFEM strategy

The strategy follows:

1. Write the **weak formulation**;
2. Apply an accurate **Stokes (Green) identity** (such that $\boldsymbol{u}$ "appears");
3. Project on a finite-dimensional space thanks to **FEM**.

Conservative System: PFEM strategy

The strategy follows:

1. Write the **weak formulation**;
2. Apply an accurate **Stokes (Green) identity** (such that $\boldsymbol{u}$ "appears");
3. Project on a finite-dimensional space thanks to **FEM**.

For all test functions $\overrightarrow{\boldsymbol{v}}_q$, v_p and v_∂ (smooth enough):

$$\begin{cases} \left\langle \partial_t \overrightarrow{\boldsymbol{\alpha}}_q, \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2} = \left\langle \overrightarrow{\mathbf{grad}}\left(e_p\right), \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2}, \\ \left\langle \partial_t \alpha_p, v_p \right\rangle_{L^2} = \left\langle \mathrm{div}\left(\overrightarrow{\boldsymbol{e}}_q\right), v_p \right\rangle_{L^2}, \\ \left\langle \boldsymbol{y}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} = \left\langle \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}. \end{cases}$$

Conservative System: PFEM strategy

The strategy follows:

1. Write the **weak formulation**;
2. Apply an accurate **Stokes (Green) identity** (such that $\boldsymbol{u}$ "appears");
3. Project on a finite-dimensional space thanks to **FEM**.

For all test functions $\overrightarrow{\boldsymbol{v}}_q$, v_p and v_∂ (smooth enough):

$$\begin{cases} \left\langle \partial_t \overrightarrow{\boldsymbol{\alpha}}_q, \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2} = \left\langle \overrightarrow{\mathbf{grad}} \left(e_p \right), \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2}, \\ \left\langle \partial_t \alpha_p, v_p \right\rangle_{L^2} = \left\langle \mathrm{div} \left(\overrightarrow{\boldsymbol{e}}_q \right), v_p \right\rangle_{L^2}, \\ \left\langle \boldsymbol{y}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} = \left\langle \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}. \end{cases}$$

Applying Green's formula on the 1st line and using the definition of $\boldsymbol{u}$:

$$\left\langle \partial_t \overrightarrow{\boldsymbol{\alpha}}_q, \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2} = - \left\langle e_p, \mathrm{div} \left(\overrightarrow{\boldsymbol{v}}_q \right) \right\rangle_{L^2} + \left\langle \overrightarrow{\boldsymbol{v}}_q \cdot \overrightarrow{\boldsymbol{n}}, \boldsymbol{u} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}.$$

Conservative System: PFEM strategy

The strategy follows:

1. Write the **weak formulation**;
2. Apply an accurate **Stokes (Green) identity** (such that $\boldsymbol{u}$ "appears");
3. Project on a finite-dimensional space thanks to **FEM**.

For all test functions $\overrightarrow{\boldsymbol{v}}_q$, v_p and v_∂ (smooth enough):

$$\begin{cases} \left\langle \partial_t \overrightarrow{\boldsymbol{\alpha}}_q, \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2} = \left\langle \overrightarrow{\mathbf{grad}}\left(e_p\right), \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2}, \\ \left\langle \partial_t \alpha_p, v_p \right\rangle_{L^2} = \left\langle \mathrm{div}\left(\overrightarrow{\boldsymbol{e}}_q\right), v_p \right\rangle_{L^2}, \\ \left\langle \boldsymbol{y}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} = \left\langle \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}. \end{cases}$$

Applying Green's formula on the 1st line and using the definition of $\boldsymbol{u}$:

$$\left\langle \partial_t \overrightarrow{\boldsymbol{\alpha}}_q, \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2} = -\left\langle e_p, \mathrm{div}\left(\overrightarrow{\boldsymbol{v}}_q\right) \right\rangle_{L^2} + \left\langle \overrightarrow{\boldsymbol{v}}_q \cdot \overrightarrow{\boldsymbol{n}}, \boldsymbol{u} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}.$$

Green's formula applied on the 2nd line would lead to normal stress control $\boldsymbol{u} = \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}$. The energy variables are **partitioned** accordingly.

Conservative System: FEM Application

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}}) \alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t),$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix,

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}}) \alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t), \quad \overrightarrow{\boldsymbol{e}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{e}_q(t),$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix,

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}})\alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t), \quad \overrightarrow{\boldsymbol{e}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{e}_q(t),$$
$$\alpha_p^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \sum_{k=1}^{N_p} \varphi_p^{k}(\overrightarrow{\boldsymbol{x}})\alpha_p^{k}(t) = \boldsymbol{\phi}_p^{\top} \cdot \underline{\alpha}_p(t), \qquad e_p^{ap}(t,\overrightarrow{\boldsymbol{x}}) = \boldsymbol{\phi}_p^{\top} \cdot \underline{e}_p(t),$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix, $\boldsymbol{\phi}_p$ an $N_p \times 1$ matrix

Conservative System: FEM Application

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}})\alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t), \quad \overrightarrow{\boldsymbol{e}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{e}_q(t),$$
$$\alpha_p^{ap}(t, \overrightarrow{\boldsymbol{x}}) := \sum_{k=1}^{N_p} \varphi_p^{k}(\overrightarrow{\boldsymbol{x}})\alpha_p^{k}(t) = \boldsymbol{\phi}_p^{\top} \cdot \underline{\alpha}_p(t), \quad e_p^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \boldsymbol{\phi}_p^{\top} \cdot \underline{e}_p(t),$$
$$\boldsymbol{u}^{ap}(t, \overrightarrow{\boldsymbol{s}}) := \sum_{m=1}^{N_\partial} \psi^{m}(\overrightarrow{\boldsymbol{s}})\boldsymbol{u}^{m}(t) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{u}}(t), \quad \boldsymbol{y}^{ap}(t, \overrightarrow{\boldsymbol{s}}) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{y}}(t),$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix, $\boldsymbol{\phi}_p$ an $N_p \times 1$ matrix and $\boldsymbol{\Psi}$ an $N_\partial \times 1$ matrix.

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$
\begin{aligned}
&\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}}) \alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t), \quad && \overrightarrow{\boldsymbol{e}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{e}_q(t),\\
&\alpha_p^{ap}(t, \overrightarrow{\boldsymbol{x}}) := \sum_{k=1}^{N_p} \varphi_p^{k}(\overrightarrow{\boldsymbol{x}}) \alpha_p^{k}(t) = \boldsymbol{\phi}_p^{\top} \cdot \underline{\alpha}_p(t), && e_p^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \boldsymbol{\phi}_p^{\top} \cdot \underline{e}_p(t),\\
&\boldsymbol{u}^{ap}(t, \overrightarrow{\boldsymbol{s}}) := \sum_{m=1}^{N_\partial} \psi^{m}(\overrightarrow{\boldsymbol{s}}) \boldsymbol{u}^{m}(t) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{u}}(t), && \boldsymbol{y}^{ap}(t, \overrightarrow{\boldsymbol{s}}) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{y}}(t),
\end{aligned}
$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix, $\boldsymbol{\phi}_p$ an $N_p \times 1$ matrix and $\boldsymbol{\Psi}$ an $N_\partial \times 1$ matrix.

The discretized system (giving the structure) then reads:

$$
\left\{
\begin{aligned}
&\overrightarrow{\boldsymbol{M}}_q \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} \underline{\alpha}_q(t) = D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t),\\
&M_p \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} \underline{\alpha}_p(t) = -D^{\top} \cdot \underline{e}_q(t),\\
&M_\partial \cdot \underline{\boldsymbol{y}}(t) = B^{\top} \cdot \underline{e}_q(t),
\end{aligned}
\right.
$$

Conservative System: FEM Application

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$
\begin{aligned}
&\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}})\alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t), \quad && \overrightarrow{\boldsymbol{e}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{e}_q(t),\\
&\alpha_p^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \sum_{k=1}^{N_p} \varphi_p^{k}(\overrightarrow{\boldsymbol{x}})\alpha_p^{k}(t) = \boldsymbol{\phi}_p^{\top} \cdot \underline{\alpha}_p(t), && e_p^{ap}(t,\overrightarrow{\boldsymbol{x}}) = \boldsymbol{\phi}_p^{\top} \cdot \underline{e}_p(t),\\
&\boldsymbol{u}^{ap}(t,\overrightarrow{\boldsymbol{s}}) := \sum_{m=1}^{N_\partial} \psi^{m}(\overrightarrow{\boldsymbol{s}})\boldsymbol{u}^{m}(t) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{u}}(t), && \boldsymbol{y}^{ap}(t,\overrightarrow{\boldsymbol{s}}) = \boldsymbol{\Psi}^{\top} \cdot \underline{\boldsymbol{y}}(t),
\end{aligned}
$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix, $\boldsymbol{\phi}_p$ an $N_p \times 1$ matrix and $\boldsymbol{\Psi}$ an $N_\partial \times 1$ matrix.

The discretized system (giving the structure) then reads:

$$
\left\{
\begin{aligned}
&\overrightarrow{\boldsymbol{M}}_q \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) = D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t),\\
&M_p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) = -D^{\top} \cdot \underline{e}_q(t),\\
&M_\partial \cdot \underline{\boldsymbol{y}}(t) = B^{\top} \cdot \underline{e}_q(t),
\end{aligned}
\right.
$$

where:

$$
\overrightarrow{\boldsymbol{M}}_q := \int_\Omega \overrightarrow{\boldsymbol{\Phi}}_q \cdot \overrightarrow{\boldsymbol{\Phi}}_q^{\top}, \qquad M_p := \int_\Omega \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^{\top}, \qquad M_\partial := \int_\Omega \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^{\top},
$$

$$
D := -\int_\Omega \mathrm{div}\left(\overrightarrow{\boldsymbol{\Phi}}_q\right) \cdot \boldsymbol{\phi}_p^{\top}, \qquad B := \int_{\partial\Omega} \left(\overrightarrow{\boldsymbol{\Phi}}_q \cdot \overrightarrow{\boldsymbol{n}}\right) \cdot \boldsymbol{\Psi}^{\top}.
$$

Conservative System: Power Balance

Finite-Dimensional Dirac Structure

$$\mathcal{J}_d := \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{D}_d := \mathrm{Graph}\left(\mathcal{J}_d\right).$$

Conservative System: Power Balance

Finite-Dimensional Dirac Structure

$$\mathcal{J}_d := \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{D}_d := \text{Graph}\left(\mathcal{J}_d\right).$$

⚠ The inner product on $\mathbb{R}^{N_q}$, $\mathbb{R}^{N_p}$ and $\mathbb{R}^{N_\partial}$ has to be taken w.r.t. the mass matrices $\overrightarrow{\boldsymbol{M}}_q$, M_p and M_∂: *e.g.* $\left\langle \overrightarrow{\boldsymbol{v}}_1, \overrightarrow{\boldsymbol{v}}_2 \right\rangle_{N_q} := \overrightarrow{\boldsymbol{v}}_2^\top \cdot \overrightarrow{\boldsymbol{M}}_q \cdot \overrightarrow{\boldsymbol{v}}_1$.

Conservative System: Power Balance

Finite-Dimensional Dirac Structure

$$\mathcal{J}_d := \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{D}_d := \mathrm{Graph}\left(\mathcal{J}_d\right).$$

⚠ The inner product on $\mathbb{R}^{N_q}$, $\mathbb{R}^{N_p}$ and $\mathbb{R}^{N_\partial}$ has to be taken w.r.t. the mass matrices $\overrightarrow{\boldsymbol{M}}_q$, M_p and M_∂: *e.g.* $\left\langle \overrightarrow{\boldsymbol{v}}_1, \overrightarrow{\boldsymbol{v}}_2 \right\rangle_{N_q} := \overrightarrow{\boldsymbol{v}}_2^\top \cdot \overrightarrow{\boldsymbol{M}}_q \cdot \overrightarrow{\boldsymbol{v}}_1$.

Discrete Hamiltonian

$$\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) := \mathcal{H}\left(\overrightarrow{\boldsymbol{\alpha}}_q^{ap}, \alpha_p^{ap}\right) = \frac{1}{2}\left(\underline{\alpha}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q + \underline{\alpha}_p^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p\right),$$

Conservative System: Power Balance

Finite-Dimensional Dirac Structure

$$\mathcal{J}_d := \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{D}_d := \mathrm{Graph}\left(\mathcal{J}_d\right).$$

⚠ The inner product on $\mathbb{R}^{N_q}$, $\mathbb{R}^{N_p}$ and $\mathbb{R}^{N_\partial}$ has to be taken w.r.t. the mass matrices $\overrightarrow{\boldsymbol{M}}_q$, M_p and M_∂: *e.g.* $\left\langle \overrightarrow{\boldsymbol{v}}_1, \overrightarrow{\boldsymbol{v}}_2 \right\rangle_{N_q} := \overrightarrow{\boldsymbol{v}}_2^\top \cdot \overrightarrow{\boldsymbol{M}}_q \cdot \overrightarrow{\boldsymbol{v}}_1$.

Discrete Hamiltonian

$$\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) := \mathcal{H}\left(\overrightarrow{\boldsymbol{\alpha}}_q^{ap}, \alpha_p^{ap}\right) = \frac{1}{2}\left(\underline{\alpha}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q + \underline{\alpha}_p^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p\right),$$

$$\overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} := \int_\Omega \overrightarrow{\boldsymbol{\Phi}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\Phi}}_q^\top \qquad \& \qquad M_{\frac{1}{\rho}} := \int_\Omega \frac{1}{\rho} \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^\top.$$

Conservative System: Power Balance

Finite-Dimensional Dirac Structure

$$\mathcal{J}_d := \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{D}_d := \mathrm{Graph}\left(\mathcal{J}_d\right).$$

⚠ The inner product on $\mathbb{R}^{N_q}$, $\mathbb{R}^{N_p}$ and $\mathbb{R}^{N_\partial}$ has to be taken w.r.t. the mass matrices $\overrightarrow{\boldsymbol{M}}_q$, M_p and M_∂: *e.g.* $\left\langle \overrightarrow{\boldsymbol{v}}_1, \overrightarrow{\boldsymbol{v}}_2 \right\rangle_{N_q} := \overrightarrow{\boldsymbol{v}}_2^\top \cdot \overrightarrow{\boldsymbol{M}}_q \cdot \overrightarrow{\boldsymbol{v}}_1$.

Discrete Hamiltonian

$$\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) := \mathcal{H}\left(\overrightarrow{\boldsymbol{\alpha}}_q^{ap}, \alpha_p^{ap}\right) = \frac{1}{2}\left(\underline{\alpha}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q + \underline{\alpha}_p^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p\right),$$

$$\overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} := \int_\Omega \overrightarrow{\boldsymbol{\Phi}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\Phi}}_q^\top \qquad \& \qquad M_{\frac{1}{\rho}} := \int_\Omega \frac{1}{\rho} \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^\top.$$

Constitutive relations: $\overrightarrow{\boldsymbol{M}}_q \cdot \underline{e}_q = \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q \quad \& \quad M_p \cdot \underline{e}_p = M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p$ ✓✓

Finite-Dimensional Dirac Structure

$$\mathcal{J}_d := \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \mathcal{D}_d := \mathrm{Graph}\left(\mathcal{J}_d\right).$$

⚠ The inner product on $\mathbb{R}^{N_q}$, $\mathbb{R}^{N_p}$ and $\mathbb{R}^{N_\partial}$ has to be taken w.r.t. the mass matrices $\overrightarrow{\boldsymbol{M}}_q$, M_p and M_∂: *e.g.* $\left\langle \overrightarrow{\boldsymbol{v}}_1, \overrightarrow{\boldsymbol{v}}_2 \right\rangle_{N_q} := \overrightarrow{\boldsymbol{v}}_2^\top \cdot \overrightarrow{\boldsymbol{M}}_q \cdot \overrightarrow{\boldsymbol{v}}_1$.

Discrete Hamiltonian

$$\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) := \mathcal{H}\left(\overrightarrow{\boldsymbol{\alpha}}_q^{ap}, \alpha_p^{ap}\right) = \frac{1}{2}\left(\underline{\alpha}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q + \underline{\alpha}_p^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p\right),$$

$$\overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} := \int_\Omega \overrightarrow{\boldsymbol{\Phi}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\Phi}}_q^\top \qquad \& \qquad M_{\frac{1}{\rho}} := \int_\Omega \frac{1}{\rho} \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^\top.$$

Constitutive relations: $\overrightarrow{\boldsymbol{M}}_q \cdot \underline{e}_q = \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q$ & $M_p \cdot \underline{e}_p = M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p$ ✓✓

Denote $\underline{f} := \left(\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q, \quad \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p, \quad -\underline{\boldsymbol{y}}\right)^\top$ and $\underline{e} := \left(\underline{e}_q, \quad \underline{e}_p, \quad \underline{\boldsymbol{u}}\right)^\top$, then:

Discrete Lossless **Power Balance**

$$\begin{pmatrix} \underline{f} \\ \underline{e} \end{pmatrix} \in \mathcal{D}_d \;\Rightarrow\; \left\langle \underline{f}, \underline{e} \right\rangle_{N_p, N_q, N_\partial} = 0 \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = \underline{\boldsymbol{u}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

Outline

Internal Dissipation: Dissipative Ports

The Hamiltonian is always the total energy:

$$\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2} \int_\Omega \left(\overrightarrow{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q + \frac{1}{\rho} \alpha_p^2 \right).$$

Internal dissipation $\epsilon(\overrightarrow{\boldsymbol{x}})\partial_t w(t, \overrightarrow{\boldsymbol{x}}) = \epsilon(\overrightarrow{\boldsymbol{x}}) e_p(t, \overrightarrow{\boldsymbol{x}})$ is added, with $\epsilon \geq 0$:

$$\left\{ \begin{array}{rl} \partial_t \overrightarrow{\boldsymbol{\alpha}}_q & = \overrightarrow{\mathbf{grad}}\left(e_p\right), \\ \partial_t \alpha_p & = \mathrm{div}\left(\overrightarrow{\boldsymbol{e}}_q\right) - \epsilon e_p, \end{array} \right. \qquad \left\{ \begin{array}{rl} \boldsymbol{u} & = e_p, \\ \boldsymbol{y} & = \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}. \end{array} \right.$$

Internal Dissipation: Dissipative Ports

The Hamiltonian is always the total energy:

$$\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2} \int_\Omega \left(\overrightarrow{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q + \frac{1}{\rho} \alpha_p^2 \right).$$

Internal dissipation $\epsilon(\overrightarrow{\boldsymbol{x}})\partial_t w(t, \overrightarrow{\boldsymbol{x}}) = \epsilon(\overrightarrow{\boldsymbol{x}}) e_p(t, \overrightarrow{\boldsymbol{x}})$ is added, with $\epsilon \geq 0$:

$$\begin{cases} \partial_t \overrightarrow{\boldsymbol{\alpha}}_q & = \overrightarrow{\mathbf{grad}}\left(e_p\right), \\ \partial_t \alpha_p & = \operatorname{div}\left(\overrightarrow{\boldsymbol{e}}_q\right) - \epsilon e_p, \end{cases} \qquad \begin{cases} \boldsymbol{u} & = e_p, \\ \boldsymbol{y} & = \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}. \end{cases}$$

$$\begin{pmatrix} \partial_t \overrightarrow{\boldsymbol{\alpha}}_q \\ \partial_t \alpha_p \end{pmatrix} = \begin{pmatrix} 0 & \overrightarrow{\mathbf{grad}} \\ \operatorname{div} & -\epsilon \end{pmatrix} \begin{pmatrix} \overrightarrow{\boldsymbol{e}}_q \\ e_p \end{pmatrix} \rightsquigarrow J := \begin{pmatrix} 0 & \overrightarrow{\mathbf{grad}} \\ \operatorname{div} & 0 \end{pmatrix}, \quad R := \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}.$$

The Hamiltonian is always the total energy:

$$\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2} \int_\Omega \left(\overrightarrow{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{\alpha}}_q + \frac{1}{\rho} \alpha_p^2 \right).$$

Internal dissipation $\epsilon(\overrightarrow{\boldsymbol{x}})\partial_t w(t, \overrightarrow{\boldsymbol{x}}) = \epsilon(\overrightarrow{\boldsymbol{x}}) e_p(t, \overrightarrow{\boldsymbol{x}})$ is added, with $\epsilon \geq 0$:

$$\begin{cases} \partial_t \overrightarrow{\boldsymbol{\alpha}}_q &= \overrightarrow{\mathbf{grad}}\left(e_p\right), \\ \partial_t \alpha_p &= \mathrm{div}\left(\overrightarrow{\boldsymbol{e}}_q\right) - \epsilon e_p, \end{cases} \qquad \begin{cases} \boldsymbol{u} &= e_p, \\ \boldsymbol{y} &= \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}. \end{cases}$$

$$\begin{pmatrix} \partial_t \overrightarrow{\boldsymbol{\alpha}}_q \\ \partial_t \alpha_p \end{pmatrix} = \begin{pmatrix} 0 & \overrightarrow{\mathbf{grad}} \\ \mathrm{div} & -\epsilon \end{pmatrix} \begin{pmatrix} \overrightarrow{\boldsymbol{e}}_q \\ e_p \end{pmatrix} \rightsquigarrow J := \begin{pmatrix} 0 & \overrightarrow{\mathbf{grad}} \\ \mathrm{div} & 0 \end{pmatrix}, \quad R := \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}.$$

Adding **dissipative ports** f_r and e_r and a **dissipative constitutive relation**:

$$\underset{\oplus\ e_r = \epsilon f_r}{\Longrightarrow} \qquad \begin{pmatrix} \partial_t \overrightarrow{\boldsymbol{\alpha}}_q \\ \partial_t \alpha_p \\ f_r \end{pmatrix} = \begin{pmatrix} 0 & \overrightarrow{\mathbf{grad}} & 0 \\ \mathrm{div} & 0 & -I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\boldsymbol{e}}_q \\ e_p \\ e_r \end{pmatrix}.$$

Internal Dissipation: Dissipative Ports

The Hamiltonian is always the total energy:

$$\mathcal{H}(\vec{\boldsymbol{\alpha}}_q, \alpha_p) := \frac{1}{2}\int_\Omega \left(\vec{\boldsymbol{\alpha}}_q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \vec{\boldsymbol{\alpha}}_q + \frac{1}{\rho}\alpha_p^2 \right).$$

Internal dissipation $\epsilon(\vec{\boldsymbol{x}})\partial_t w(t, \vec{\boldsymbol{x}}) = \epsilon(\vec{\boldsymbol{x}}) e_p(t, \vec{\boldsymbol{x}})$ is added, with $\epsilon \geq 0$:

$$\begin{cases} \partial_t \vec{\boldsymbol{\alpha}}_q & = \overrightarrow{\mathbf{grad}}\,(e_p)\,, \\ \partial_t \alpha_p & = \mathrm{div}\,\left(\vec{\boldsymbol{e}}_q\right) - \epsilon e_p, \end{cases} \qquad \begin{cases} \boldsymbol{u} & = e_p, \\ \boldsymbol{y} & = \vec{\boldsymbol{e}}_q \cdot \vec{\boldsymbol{n}}. \end{cases}$$

$$\begin{pmatrix} \partial_t \vec{\boldsymbol{\alpha}}_q \\ \partial_t \alpha_p \end{pmatrix} = \begin{pmatrix} 0 & \overrightarrow{\mathbf{grad}} \\ \mathrm{div} & -\epsilon \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{e}}_q \\ e_p \end{pmatrix} \rightsquigarrow J := \begin{pmatrix} 0 & \overrightarrow{\mathbf{grad}} \\ \mathrm{div} & 0 \end{pmatrix}, \quad R := \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}.$$

Adding **dissipative ports** f_r and e_r and a **dissipative constitutive relation**:

$$\underset{\oplus\; e_r = \epsilon f_r}{\Longrightarrow} \qquad \begin{pmatrix} \partial_t \vec{\boldsymbol{\alpha}}_q \\ \partial_t \alpha_p \\ f_r \end{pmatrix} = \begin{pmatrix} 0 & \overrightarrow{\mathbf{grad}} & 0 \\ \mathrm{div} & 0 & -I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{e}}_q \\ e_p \\ e_r \end{pmatrix}.$$

Lossy Power Balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\vec{\boldsymbol{\alpha}}_q, \alpha_p) = -\left\langle \epsilon e_p, e_p \right\rangle_{L^2} + \left\langle \boldsymbol{y}, \boldsymbol{u} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \leq \left\langle \boldsymbol{y}, \boldsymbol{u} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}.$$

Internal Dissipation: PFEM

Approximating f_r and e_r in the FEM basis ϕ_p, PFEM gives:

$$\underbrace{\begin{pmatrix} \overrightarrow{\boldsymbol{M}}_q & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 \\ 0 & 0 & M_p & 0 \\ 0 & 0 & 0 & M_\partial \end{pmatrix}}_{\mathcal{M}} \underbrace{\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) \\ \underline{f}_r(t) \\ -\underline{\boldsymbol{y}}(t) \end{pmatrix}}_{\overrightarrow{\boldsymbol{f}}_d} = \underbrace{\begin{pmatrix} 0 & D & 0 & B \\ -D^\top & 0 & M_p & 0 \\ 0 & -M_p & 0 & 0 \\ -B^\top & 0 & 0 & 0 \end{pmatrix}}_{\mathcal{J}_d} \underbrace{\begin{pmatrix} \underline{e}_q(t) \\ \underline{e}_p(t) \\ \underline{e}_r(t) \\ \underline{\boldsymbol{u}}(t) \end{pmatrix}}_{\overrightarrow{\boldsymbol{e}}_d}.$$

Internal Dissipation: PFEM

Approximating f_r and e_r in the FEM basis ϕ_p, PFEM gives:

$$\underbrace{\begin{pmatrix} \overrightarrow{\boldsymbol{M}}_q & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 \\ 0 & 0 & M_p & 0 \\ 0 & 0 & 0 & M_\partial \end{pmatrix}}_{\mathcal{M}} \underbrace{\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) \\ \underline{f}_r(t) \\ -\underline{\boldsymbol{y}}(t) \end{pmatrix}}_{\overrightarrow{\boldsymbol{f}}_d} = \underbrace{\begin{pmatrix} 0 & D & 0 & B \\ -D^\top & 0 & M_p & 0 \\ 0 & -M_p & 0 & 0 \\ -B^\top & 0 & 0 & 0 \end{pmatrix}}_{\mathcal{J}_d} \underbrace{\begin{pmatrix} \underline{e}_q(t) \\ \underline{e}_p(t) \\ \underline{e}_r(t) \\ \underline{\boldsymbol{u}}(t) \end{pmatrix}}_{\overrightarrow{\boldsymbol{e}}_d}.$$

The **dissipative constitutive relation** is discretized as:

$$M_p \cdot \underline{e}_r = E \cdot \underline{f}_r, \qquad \text{with } E := \int_\Omega \epsilon \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^\top \geq 0.$$

Approximating f_r and e_r in the FEM basis ϕ_p, PFEM gives:

$$\underbrace{\begin{pmatrix} \overrightarrow{\boldsymbol{M}}_q & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 \\ 0 & 0 & M_p & 0 \\ 0 & 0 & 0 & M_\partial \end{pmatrix}}_{\mathcal{M}} \underbrace{\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) \\ \underline{f}_r(t) \\ -\underline{\boldsymbol{y}}(t) \end{pmatrix}}_{\overrightarrow{\boldsymbol{f}}_d} = \underbrace{\begin{pmatrix} 0 & D & 0 & B \\ -D^\top & 0 & M_p & 0 \\ 0 & -M_p & 0 & 0 \\ -B^\top & 0 & 0 & 0 \end{pmatrix}}_{\mathcal{J}_d} \underbrace{\begin{pmatrix} \underline{e}_q(t) \\ \underline{e}_p(t) \\ \underline{e}_r(t) \\ \underline{\boldsymbol{u}}(t) \end{pmatrix}}_{\overrightarrow{\boldsymbol{e}}_d}.$$

The **dissipative constitutive relation** is discretized as:

$$M_p \cdot \underline{e}_r = E \cdot \underline{f}_r, \qquad \text{with } E := \int_\Omega \epsilon \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^\top \geq 0.$$

The **extended Dirac structure** $\mathcal{D}_d^\epsilon := \mathrm{Graph}\,(\mathcal{J}_d)$, w.r.t. the $\mathcal{M}$-weighted scalar product in $\mathbb{R}^{N_q+2N_p+N_\partial}$, takes into account for any $\epsilon \geq 0$.

Approximating f_r and e_r in the FEM basis ϕ_p, PFEM gives:

$$\underbrace{\begin{pmatrix} \overrightarrow{\boldsymbol{M}}_q & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 \\ 0 & 0 & M_p & 0 \\ 0 & 0 & 0 & M_\partial \end{pmatrix}}_{\mathcal{M}} \underbrace{\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) \\ \underline{f}_r(t) \\ -\underline{\boldsymbol{y}}(t) \end{pmatrix}}_{\overrightarrow{\boldsymbol{f}}_d} = \underbrace{\begin{pmatrix} 0 & D & 0 & B \\ -D^\top & 0 & M_p & 0 \\ 0 & -M_p & 0 & 0 \\ -B^\top & 0 & 0 & 0 \end{pmatrix}}_{\mathcal{J}_d} \underbrace{\begin{pmatrix} \underline{e}_q(t) \\ \underline{e}_p(t) \\ \underline{e}_r(t) \\ \underline{\boldsymbol{u}}(t) \end{pmatrix}}_{\overrightarrow{\boldsymbol{e}}_d}.$$

The **dissipative constitutive relation** is discretized as:

$$M_p \cdot \underline{e}_r = E \cdot \underline{f}_r, \qquad \text{with } E := \int_\Omega \epsilon \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^\top \geq 0.$$

The **extended Dirac structure** $\mathcal{D}_d^\epsilon := \mathrm{Graph}\,(\mathcal{J}_d)$, w.r.t. the $\mathcal{M}$-weighted scalar product in $\mathbb{R}^{N_q+2N_p+N_\partial}$, takes into account for any $\epsilon \geq 0$.

Discrete Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = -\underline{e}_p^\top \cdot E \cdot \underline{e}_p + \underline{\boldsymbol{u}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}} \leq \underline{\boldsymbol{u}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

⚠ In practice, f_r and e_r do not need to be discretized in the basis of f_p and e_p.

Outline

Boundary Dissipation: Impedance Ports

The Impedance Boundary Condition (IBC), with $Z \geq 0$ on $\partial\Omega$, and $\boldsymbol{\nu}$ as new control, is considered: $\boldsymbol{\nu} = e_p + Z \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}} \Leftrightarrow \boldsymbol{\nu} = \partial_t w + Z \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}}\,(w) \right) \cdot \overrightarrow{\boldsymbol{n}}$.

Boundary Dissipation: Impedance Ports

The Impedance Boundary Condition (IBC), with $Z \geq 0$ on $\partial\Omega$, and $\boldsymbol{\nu}$ as new control, is considered: $\boldsymbol{\nu} = e_p + Z \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}} \Leftrightarrow \boldsymbol{\nu} = \partial_t w + Z \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}}(w) \right) \cdot \overrightarrow{\boldsymbol{n}}$.

⚠ This kind of dissipation does not *easily* fit in the "$J - R$ framework".

Boundary Dissipation: Impedance Ports

The Impedance Boundary Condition (IBC), with $Z \geq 0$ on $\partial\Omega$, and $\boldsymbol{\nu}$ as new control, is considered: $\boldsymbol{\nu} = e_p + Z \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}} \Leftrightarrow \boldsymbol{\nu} = \partial_t w + Z \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\textbf{grad}} (w) \right) \cdot \overrightarrow{\boldsymbol{n}}$.

⚠ This kind of dissipation does not *easily* fit in the "$J - R$ framework".
It can be seen as an *output feedback law* $\boldsymbol{u} = -Z\boldsymbol{y} + \boldsymbol{\nu}$ in the previous case.

Boundary Dissipation: Impedance Ports

The Impedance Boundary Condition (IBC), with $Z \geq 0$ on $\partial\Omega$, and $\boldsymbol{\nu}$ as new control, is considered: $\boldsymbol{\nu} = e_p + Z \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}} \Leftrightarrow \; \boldsymbol{\nu} = \partial_t w + Z \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\textbf{grad}}\,(w) \right) \cdot \overrightarrow{\boldsymbol{n}}$.

⚠ This kind of dissipation does not *easily* fit in the "$J - R$ framework".
It can be seen as an *output feedback law* $\boldsymbol{u} = -Z\boldsymbol{y} + \boldsymbol{\nu}$ in the previous case.

Lossy Power Balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) = -\left\langle \epsilon e_p, e_p \right\rangle_{L^2} - \left\langle \boldsymbol{y}, Z\boldsymbol{y} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} + \left\langle \boldsymbol{y}, \boldsymbol{\nu} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} .$$

Boundary Dissipation: Impedance Ports

The Impedance Boundary Condition (IBC), with $Z \geq 0$ on $\partial\Omega$, and $\boldsymbol{\nu}$ as new control, is considered: $\boldsymbol{\nu} = e_p + Z \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}} \Leftrightarrow \boldsymbol{\nu} = \partial_t w + Z \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}}(w) \right) \cdot \overrightarrow{\boldsymbol{n}}$.

⚠ This kind of dissipation does not *easily* fit in the "$J - R$ framework".
It can be seen as an *output feedback law* $\boldsymbol{u} = -Z\boldsymbol{y} + \boldsymbol{\nu}$ in the previous case.

Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) = -\langle \epsilon e_p, e_p \rangle_{L^2} - \langle \boldsymbol{y}, Z\boldsymbol{y} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} + \langle \boldsymbol{y}, \boldsymbol{\nu} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} .$$

Add **impedance ports** (f_i, e_i) and **dissipative constitutive relation** $e_i = Z f_i$, and approximate f_i and e_i in the FEM basis $\boldsymbol{\Psi}$, PFEM gives:

$$\begin{pmatrix} \overrightarrow{\boldsymbol{M}}_q & 0 & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 & 0 \\ 0 & 0 & M_p & 0 & 0 \\ 0 & 0 & 0 & M_\partial & 0 \\ 0 & 0 & 0 & 0 & M_\partial \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) \\ \underline{f}_r(t) \\ \underline{f}_i(t) \\ -\underline{\boldsymbol{y}}(t) \end{pmatrix} = \begin{pmatrix} 0 & D & 0 & -B & B \\ -D^\top & 0 & M_p & 0 & 0 \\ 0 & -M_p & 0 & 0 & 0 \\ B^\top & 0 & 0 & 0 & 0 \\ -B^\top & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{e}_q(t) \\ \underline{e}_p(t) \\ \underline{e}_r(t) \\ \underline{e}_i(t) \\ \underline{\boldsymbol{\nu}}(t) \end{pmatrix}$$

and $M_\partial \cdot \underline{e}_i = \langle Z \rangle \cdot \underline{f}_i$, with $\langle Z \rangle := \int_{\partial\Omega} Z \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^\top \geq 0$.

The Impedance Boundary Condition (IBC), with $Z \geq 0$ on $\partial\Omega$, and $\boldsymbol{\nu}$ as new control, is considered: $\boldsymbol{\nu} = e_p + Z \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}} \Leftrightarrow \; \boldsymbol{\nu} = \partial_t w + Z \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\textbf{grad}}\,(w) \right) \cdot \overrightarrow{\boldsymbol{n}}$.

⚠ This kind of dissipation does not *easily* fit in the "$J - R$ framework".
It can be seen as an *output feedback law* $\boldsymbol{u} = -Z\boldsymbol{y} + \boldsymbol{\nu}$ in the previous case.

Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) = -\left\langle \epsilon e_p, e_p \right\rangle_{L^2} - \left\langle \boldsymbol{y}, Z\boldsymbol{y} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} + \left\langle \boldsymbol{y}, \boldsymbol{\nu} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}.$$

Add **impedance ports** (f_i, e_i) and **dissipative constitutive relation** $e_i = Z f_i$, and approximate f_i and e_i in the FEM basis $\boldsymbol{\Psi}$, PFEM gives:

$$\begin{pmatrix} \overrightarrow{\boldsymbol{M}}_q & 0 & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 & 0 \\ 0 & 0 & M_p & 0 & 0 \\ 0 & 0 & 0 & M_\partial & 0 \\ 0 & 0 & 0 & 0 & M_\partial \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) \\ \underline{f}_r(t) \\ \underline{f}_i(t) \\ -\underline{\boldsymbol{y}}(t) \end{pmatrix} = \begin{pmatrix} 0 & D & 0 & -B & B \\ -D^\top & 0 & M_p & 0 & 0 \\ 0 & -M_p & 0 & 0 & 0 \\ B^\top & 0 & 0 & 0 & 0 \\ -B^\top & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{e}_q(t) \\ \underline{e}_p(t) \\ \underline{e}_r(t) \\ \underline{e}_i(t) \\ \underline{\boldsymbol{\nu}}(t) \end{pmatrix}$$

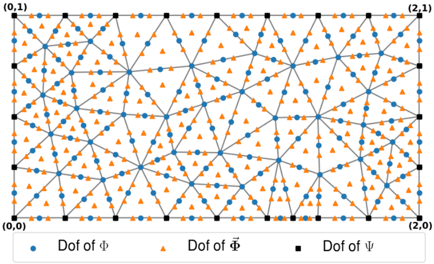
and $M_\partial \cdot \underline{e}_i = \langle Z \rangle \cdot \underline{f}_i$, with $\langle Z \rangle := \int_{\partial\Omega} Z \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^\top \geq 0$.

Discrete Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = -\underline{e}_p^\top \cdot E \cdot \underline{e}_p - \underline{\boldsymbol{y}}^\top \cdot \langle Z \rangle \cdot \underline{\boldsymbol{y}} + \underline{\boldsymbol{\nu}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

Boundary Dissipation: Simulations

- Heteregenous ($\rho \not\equiv constant$);
- Anisotropic (tensor $\overline{\overline{T}} \not\equiv constant$);
- $\epsilon \equiv 0$;
- $Z \neq 0$ for $t \geq 2$;
- Raviart-Thomas FEM for q-variables;
- Lagrange FEM for p-variables;
- Lagrange FEM for ∂-variables;



Outline

Conclusion and Further Works

To sum up:

A structure-preserving method has been proposed for dissipative port-Hamiltonian Systems, with the following strategy:

- **Add ports** to get a Dirac structure;
- Write down **weak formulations**;
- Apply **Stokes formula on a Partition** of the system;
- Apply the **Finite Elements Method**;

Furthermore: *diffusion model* as **heat equation** can be handled.

To go further:

- Choice for the **finite elements families**:
  - Convergence rate?
  - *Conformity:* $\mathcal{D}_d \subset \mathcal{D}$?
- **Mixed** boundary control;
- **Symplectic** time-integration? ⚠ *DAE*!!!

References

A structure-preserving Partitioned Finite Element Method for the 2D wave equation
Cardoso-Ribeiro F.L., Matignon D., Lefèvre L.
IFAC-PapersOnLine, vol.51(3), pp.119–124 (2018), LHMNC 2018

Structure preserving approximation of dissipative evolution problems
Egger H.
Numerische Mathematik, vol.143(1), pp.85–106 (2019)

Energy-Preserving and Passivity-Consistent Numerical Discretization of Port-Hamiltonian Systems
Celledoni E., Høiseth, E.H.
arXiv:1706.08621, (2017)

Hamiltonian formulation of distributed-parameter systems with boundary energy flow
van der Schaft A. J., Maschke B.
Journal of Geometry and Physics, vol.42(1–2), pp.166–194 (2002)

Thank you for your attention!

- **Space domain and physical parameters:**
  - $\Omega \subset \mathbb{R}^{n\geq 1}$ is a bounded open connected set;
  - $\overrightarrow{\boldsymbol{n}}$ is the outward unit normal on the boundary $\partial\Omega$;
  - $\rho(\overrightarrow{\boldsymbol{x}})$ is the mass density;
  - $\overline{\overline{\boldsymbol{T}}}(\overrightarrow{\boldsymbol{x}})$ is the conductivity tensor.

- **Notations:**
  - T is the local temperature;
  - $\beta := \frac{1}{T}$ is the reciprocal temperature;
  - u is the internal energy density;
  - s is the entropy density;
  - $\overrightarrow{\boldsymbol{J}}_Q$ is the heat flux;
  - $\overrightarrow{\boldsymbol{J}}_S := \beta \overrightarrow{\boldsymbol{J}}_Q$ is the entropy flux;
  - $C_V := \left(\frac{\mathrm{d}u}{\mathrm{d}T}\right)_V$ is the isochoric heat capacity.

- **"Context & Axioms":**
  - **Medium:** rigid body without chemical reaction;
  - **1st law of thermodynamics:**

$$\rho(\overrightarrow{\boldsymbol{x}})\partial_t u(t, \overrightarrow{\boldsymbol{x}}) = -\mathrm{div}\left(\overrightarrow{\boldsymbol{J}}_Q(t, \overrightarrow{\boldsymbol{x}})\right);$$

  - **Gibbs' relation:**

$$\mathrm{d}U = T\ \mathrm{d}S, \quad \Longrightarrow \quad \partial_t u(t, \overrightarrow{\boldsymbol{x}}) = T(t, \overrightarrow{\boldsymbol{x}})\partial_t s(t, \overrightarrow{\boldsymbol{x}});$$

  - **Entropy evolution:**

$$\rho(\overrightarrow{\boldsymbol{x}})\partial_t s(t, \overrightarrow{\boldsymbol{x}}) = -\mathrm{div}\left(\overrightarrow{\boldsymbol{J}}_S(t, \overrightarrow{\boldsymbol{x}})\right) + \sigma(t, \overrightarrow{\boldsymbol{x}}),$$

  with $\sigma := \overrightarrow{\mathbf{grad}}\,(\beta) \cdot \overrightarrow{\boldsymbol{J}}_Q$ is the *irreversible entropy production*.

- **"Laws":**
  - **Fourier's law:**

$$\overrightarrow{\boldsymbol{J}}_Q(t, \overrightarrow{\boldsymbol{x}}) = -\overline{\overline{\boldsymbol{T}}}(t, \overrightarrow{\boldsymbol{x}}) \cdot \overrightarrow{\mathbf{grad}}\left(T(t, \overrightarrow{\boldsymbol{x}})\right);$$

  - **Dulong-Petit's law:**

$$u(t, \overrightarrow{\boldsymbol{x}}) = C_V(\overrightarrow{\boldsymbol{x}})T(t, \overrightarrow{\boldsymbol{x}}).$$

Diffusion: Lyapunov Functional

Quadratic Hamiltonian: Lyapunov Functional

$$\mathcal{H}(u(t,\vec{x})) := \frac{1}{2}\int_\Omega \rho(\vec{x})\frac{\left(u(t,\vec{x})\right)^2}{C_V(t,\vec{x})}\,\mathrm{d}\vec{x},$$

$\alpha_u := u$ is the **energy variable**, and $e_u := \delta^\rho_{\alpha_u} = \frac{u}{C_V}$ the **co-energy variable**.

Under *Dulong-Petit's law*, this is the *usual* functional used in the mathmatics community: $\mathcal{H} := \int_\Omega \rho C_v T^2$, even if **its physical meaning is far to be clear**.

Power Balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = \int_\Omega \vec{J}_Q \cdot \overrightarrow{\mathbf{grad}}\left(\frac{u}{C_V}\right) - \int_{\partial\Omega}\frac{u}{C_V}\vec{J}_Q\cdot\vec{n} - \frac{1}{2}\int_\Omega \rho\partial_t C_V\left(\frac{u}{C_V}\right)^2.$$

Defining $f_u := \partial_t\alpha_u = \partial_t u$, $e_u = \frac{u}{C_V}$, $\vec{f}_Q := -\overrightarrow{\mathbf{grad}}\left(\frac{u}{C_V}\right)$, and $\vec{e}_Q := \vec{J}_Q$:

$$\begin{pmatrix}\rho f_u \\ \vec{f}_Q\end{pmatrix} = \begin{pmatrix}0 & -\mathrm{div} \\ -\overrightarrow{\mathbf{grad}} & 0\end{pmatrix}\begin{pmatrix}e_u \\ \vec{e}_Q\end{pmatrix}.$$

Diffusion: Lyapunov Functional

At least two choices for boundary control: e_u or $\overrightarrow{\boldsymbol{e}}_Q \cdot \overrightarrow{\boldsymbol{n}}$.

With inward flux control $\boldsymbol{\nu} = -\overrightarrow{\boldsymbol{e}}_Q \cdot \overrightarrow{\boldsymbol{n}}$, the output is $\boldsymbol{y} = \overrightarrow{\boldsymbol{e}}_u$, *i.e.* the boundary temperature using Dulong-Petit's law, and the discretized system is:

$$\begin{pmatrix} M_\rho & 0 & 0 \\ 0 & \overrightarrow{\boldsymbol{M}} & 0 \\ 0 & 0 & M_\partial \end{pmatrix} \begin{pmatrix} \underline{f}_u \\ \underline{f}_Q \\ -\underline{\boldsymbol{y}} \end{pmatrix} = \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{e}_u \\ \underline{e}_Q \\ \underline{\boldsymbol{\nu}} \end{pmatrix},$$

& **constitutive relations:** $M_{\rho C_V} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \underline{e}_u = M_\rho \cdot \underline{f}_u$ & $\overrightarrow{\boldsymbol{M}} \cdot \underline{\overrightarrow{\boldsymbol{e}}}_Q = \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{f}_Q$.

Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} := -\int_\Omega \overrightarrow{\boldsymbol{f}}_Q \cdot \overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{f}}_Q + \int_{\partial\Omega} \boldsymbol{y}\boldsymbol{\nu}.$$

Discrete Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} := -\underline{f}_Q \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{f}_Q + \underline{\boldsymbol{\nu}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

Diffusion: Internal Energy

Let us take as Hamiltonian the internal energy in function of the entropy:

$$\mathcal{U}(s(t, \overrightarrow{\boldsymbol{x}})) := \int_\Omega \rho(\overrightarrow{\boldsymbol{x}}) u(s(t, \overrightarrow{\boldsymbol{x}})) \, \mathrm{d}\overrightarrow{\boldsymbol{x}},$$

together with $\boldsymbol{\nu} = T$ and $\boldsymbol{y} = \overrightarrow{\boldsymbol{J}}_S \cdot \overrightarrow{\boldsymbol{n}}$.

Power Balance (first law of thermodynamics)

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{U}(s) = \langle \boldsymbol{y}, \boldsymbol{\nu} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} .$$

Adding *entropy ports* with the **entropy constitutive relation** (definition of σ): $T\sigma = -\overrightarrow{\mathbf{grad}}\,(T) \cdot \overrightarrow{\boldsymbol{J}}_S$, leads to a PHDAE.

Gibbs' relation is a first constitutive relation, and Fourier's law can be the other.

Discrete **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{U}_d(\overline{s}) = \boldsymbol{\nu}^\perp \cdot M_\partial \cdot \boldsymbol{y}.$$

Diffusion: Simulations

Institut Supérieur de l'Aéronautique et de l'Espace

10 avenue Édouard Belin – BP 54032

31055 Toulouse Cedex 4 – France

Phone: +33 5 61 33 80 80

www.isae-supaero.fr