Partitioned Finite Element Method for Port-Hamiltonian Systems with Boundary Damping:
Anistropic Heterogeneous 2D Wave Equations

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Overview

Waves with impedance boundary condition (IBC)

Anisotropic Heterogeneous Wave equation with boundary damping ($\Omega \underset{bounded}{\overset{open}{\subset}} \mathbb{R}^2$):

$$\left\{\begin{array}{l} \rho(\mathbf{x})\,\partial^2_{tt} w(t,\mathbf{x}) = \mathrm{div}\left(\overline{\overline{T}}(\mathbf{x})\cdot\mathbf{grad}\, w(t,\mathbf{x})\right),\ \mathbf{x}\in\Omega,\quad (PDE) \\ \underbrace{Z(\mathbf{x})}_{Impedance}\ \underbrace{\left(\overline{\overline{T}}(\mathbf{x})\cdot\mathbf{grad}\, w(t,\mathbf{x})\right)\cdot\mathbf{n}}_{Neumann\ trace} + \underbrace{\partial_t w(t,\mathbf{x})}_{Dirichlet\ trace} = 0,\ \mathbf{x}\in\partial\Omega,\quad (BC) \\ \left.\begin{array}{l} w(0,\mathbf{x}) = w_0(\mathbf{x}),\ \mathbf{x}\in\Omega,\ t=0, \\ \partial_t w(0,\mathbf{x}) = w_1(\mathbf{x}),\ \mathbf{x}\in\Omega,\ t=0, \end{array}\right. \quad (Initial\ data) \end{array}\right.$$

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- $w(t,\mathbf{x})$ deflection from equilibrium.
- $\overline{\overline{T}}(\mathbf{x})$ Young's elasticity modulus.
- $\rho(\mathbf{x})$ mass density.
- $Z(\mathbf{x})$ impedance.

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In 1D, $\overline{\overline{T}} = T_0$ and $\rho = \rho_0$, *characteristic impedance*

$$Z_c = \sqrt{T_0\rho_0}$$

Port-Hamiltonian formulation 1/3

- Introduce the *energy variables* $\boldsymbol{\alpha} := \begin{bmatrix} \boldsymbol{\alpha}_q{}^\top & \alpha_p \end{bmatrix}^\top$

$$\underset{(Strain)}{\boldsymbol{\alpha}_q := \mathbf{grad}\, w}, \qquad \underset{(Linear\ momentum)}{\alpha_p := \rho\, \partial_t w},$$

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Hamiltonian: (total mechanical energy)

$$\mathcal{H}(t) = \frac{1}{2} \int_\Omega \boldsymbol{\alpha}_q(t,\mathbf{x})^\top \cdot \overline{\overline{T}}(\mathbf{x}) \cdot \boldsymbol{\alpha}_q(t,\mathbf{x}) + \alpha_p(t,\mathbf{x}) \frac{1}{\rho(\mathbf{x})} \alpha_p(t,\mathbf{x}) \, d\mathbf{x},$$

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- The corresponding co-energy variables $\boldsymbol{e} := \begin{bmatrix} \boldsymbol{e}_q^\top & e_p \end{bmatrix}^\top$,

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Infinite-dimensional port-Hamiltonian system:

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Port-Hamiltonian formulation 2/3 - Dissipation

Power balance:

$$\frac{d}{dt}\mathcal{H}(t) = \left\langle e_p, \mathbf{e}_q \cdot \mathbf{n} \right\rangle_{\partial\Omega} = -\left\langle Z\, \mathbf{e}_q \cdot \mathbf{n}, \mathbf{e}_q \cdot \mathbf{n} \right\rangle_{\partial\Omega} = -\left\| \sqrt{Z}\, \mathbf{e}_q \cdot \mathbf{n} \right\|^2_{L^2(\partial\Omega)} \leq 0$$

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☺ At the discrete level, we will get $\partial_t \boldsymbol{\alpha}_d = (J_d - R_d)\mathbf{e}_d$!

Port-Hamiltonian formulation 3/3 - Well-posedness

Existence and uniqueness: [Kurula and Zwart, 2015]

Theorem

$$\forall(\boldsymbol{\alpha}_q^0, \alpha_p^0) \in \overline{\overline{T}}^{-1} \boldsymbol{H}^{\mathrm{div}}(\Omega) \times \rho H^1(\Omega) \quad \text{(initial data)},$$

$$\exists!(\boldsymbol{\alpha}_q, \alpha_p) \in \mathcal{C}\left(0,\infty; \overline{\overline{T}}^{-1} \boldsymbol{H}^{\mathrm{div}}(\Omega) \times \rho H^1(\Omega)\right) \cap \mathcal{C}^1\left(0,\infty; \boldsymbol{L}^2(\Omega) \times L^2(\Omega)\right),$$

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- $\rho H^1(\Omega) := \left\{v_p \in L^2(\Omega); \quad \mathbf{grad}(\rho v_p) \in L^2(\Omega)\right\}$.

Structure-Preserving Discretizations Overview

Discrete methods based on the Geometry of the system:

- Explicit simplicial discretization [Seslija et al., 2014]
- Discretization based on primal-dual complex [Kotyczka and Maschke*, 2017]
- Finite difference methods (FDM) [Trenchant et al.*, 2018]
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- Mixed finite element method [Golo et al., 2004]
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1. Use PFEM to discretize the port-Hamiltonian system.
2. Use an output-feedback law to take the impedance BC into account.

Open-Loop Discretization 1/5 - Weak formulation

Boundary controlled and observed wave equation:

$$\begin{cases} \rho \partial_{tt}^2 w = \mathrm{div}(\overline{\overline{T}}\, \mathbf{grad}\, w), \\ u_\partial = \partial_t w\big|_{\partial\Omega}, \\ y_\partial = \overline{\overline{T}}\, \mathbf{grad}\, w \cdot \mathbf{n}\big|_{\partial\Omega}. \end{cases}$$

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$$\begin{cases} \rho \partial_{tt}^2 w = \operatorname{div}(\overline{\overline{T}} \, \mathbf{grad} \, w), \\ u_\partial = \partial_t w \big|_{\partial\Omega}, \\ y_\partial = \overline{\overline{T}} \, \mathbf{grad} \, w \cdot \mathbf{n} \big|_{\partial\Omega}. \end{cases} \quad \overset{(PHS)}{\Longrightarrow} \quad \begin{cases} \partial_t \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \operatorname{div} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix}, \\ u_\partial = e_p \big|_{\partial\Omega}, \\ y_\partial = \boldsymbol{e}_q \cdot \mathbf{n} \big|_{\partial\Omega}. \end{cases}$$

Its power balance is then: $\frac{d}{dt}\mathcal{H}(t) = \langle u_\partial, y_\partial \rangle_{\partial\Omega}$.

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STEP 1: Weak form

$$\begin{cases} (\partial_t \boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega & = (\mathbf{grad}\, e_p, \boldsymbol{v}_q)_\Omega, \\ (\partial_t \alpha_p, v_p)_\Omega & = (\operatorname{div} \boldsymbol{e}_q, v_p)_\Omega, \end{cases}$$

where $\boldsymbol{v}_q$ and v_p are sufficiently smooth test functions.

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STEP 2: Green's formula with $e_p|_{\partial\Omega} = u_\partial$

$$\begin{cases} (\partial_t \boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega = -(e_p, \mathrm{div}\, \boldsymbol{v}_q)_\Omega + \langle u_\partial, \boldsymbol{v}_q \cdot \mathbf{n} \rangle_{\partial\Omega}, \\ (\partial_t \alpha_p, v_p)_\Omega = (\mathrm{div}\, \boldsymbol{e}_q, v_p)_\Omega, \\ \langle y_\partial, v_\partial \rangle_{\partial\Omega} = \langle \boldsymbol{e}_q \cdot \mathbf{n}, v_\partial \rangle_{\partial\Omega}. \end{cases} \tag{2}$$

Open-Loop Discretization 2/5 - Approximation families

Finite-dimensional basis families:

$$\mathcal{V}_q := \text{span}\{\vec{\varphi}_q^{\,i}\}_{1\leq i\leq N_q}, \quad \mathcal{V}_p := \text{span}\{\varphi_p^k\}_{1\leq k\leq N_p} \quad \text{and} \quad \mathcal{V}_\partial := \text{span}\{\psi_\partial^m\}_{1\leq m\leq N_\partial}.$$

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Approached solutions:

$$\boldsymbol{\alpha}_q(t,\mathbf{x}) \approx \Sigma_{i=1}^{N_q} \alpha_q^i(t)\vec{\boldsymbol{\varphi}}_q^i(\mathbf{x}) = \vec{\boldsymbol{\Phi}}_q^\top \cdot \underline{\alpha}_q, \qquad \mathbf{e}_q(t,\mathbf{x}) \approx \Sigma_{i=1}^{N_q} e_q^i(t)\vec{\boldsymbol{\varphi}}_q^i(\mathbf{x}) = \vec{\boldsymbol{\Phi}}_q^\top \cdot \underline{e}_q,$$

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Port-Hamiltonian system in the approximation basis:

$$\begin{cases} \Sigma_{i=1}^{N_q} \left(\vec{\varphi}_q^{\,i}, \vec{\varphi}_q^{\,j}\right)_\Omega \dfrac{d}{dt}\alpha_q^i = -\Sigma_{k=1}^{N_p}\left(\varphi_p^k, \text{div}\,\vec{\varphi}_q^{\,j}\right)_\Omega e_p^k + \Sigma_{m=1}^{N_\partial}\left\langle \psi_\partial^m, \vec{\varphi}_q^{\,j}\cdot\mathbf{n}\right\rangle_{\partial\Omega} u_\partial^m, \ j = 1,\dots,N_q \\ \Sigma_{k=1}^{N_p}\left(\varphi_p^k, \varphi_p^\ell\right)_\Omega \dfrac{d}{dt}\alpha_p^k = \Sigma_{i=1}^{N_q}\left(\text{div}\,\vec{\varphi}_q^{\,i}, \varphi_p^\ell\right)_\Omega e_q^i, \ \ell = 1,\dots,N_p \\ \Sigma_{m=1}^{N_\partial}\left\langle \psi_\partial^m, \psi_\partial^n\right\rangle_{\partial\Omega} y_\partial^m = \Sigma_{i=1}^{N_q}\left\langle \vec{\varphi}_q^{\,i}\cdot\mathbf{n}, \psi_\partial^n\right\rangle_{\partial\Omega} e_q^i, \ n = 1,\dots,N_\partial \end{cases}$$

Open-Loop Discretization 3/5 - Interconnection structure

Matrix form:

$$\begin{cases} M_q \, \dfrac{d}{dt}\underline{\alpha}_q = D \, \underline{e}_p + B\underline{u}_\partial \\ M_p \, \dfrac{d}{dt}\underline{\alpha}_p = -D^\top \, \underline{e}_q \\ \qquad M_\partial \underline{y}_\partial = B^\top \underline{e}_q \end{cases}$$

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where,

$$\begin{aligned}
(M_q)_{ij} &= (\vec{\varphi}_q^j, \vec{\varphi}_q^i)_\Omega, & M_q &= \textstyle\int_\Omega \vec{\Phi}_q \cdot \vec{\Phi}_q^\top \in \mathbb{R}^{N_q \times N_q}, \\
(M_p)_{k\ell} &= (\varphi_p^\ell, \varphi_p^k)_\Omega, & M_p &= \textstyle\int_\Omega \Phi_p \cdot \vec{\Phi}_q^\top \in \mathbb{R}^{N_p \times N_p}, \\
(D)_{jk} &= -(\varphi_p^k, \operatorname{div} \vec{\varphi}_q^j)_\Omega, & D &= -\textstyle\int_\Omega \operatorname{div} \vec{\Phi}_q \cdot \Phi_p^\top \in \mathbb{R}^{N_q \times N_p}, \\
(B)_{jm} &= (\psi_\partial^m, \vec{\varphi}_q^j \cdot \mathbf{n})_{\partial\Omega}, & B &= \textstyle\int_{\partial\Omega} \vec{\Phi}_q \cdot \mathbf{n} \cdot \Psi_\partial^\top \in \mathbb{R}^{N_q \times N_\partial}, \\
(M_\partial)_{mn} &= (\psi_\partial^n, \psi_\partial^m)_{\partial\Omega}, & M_\partial &= \textstyle\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top \in \mathbb{R}^{N_\partial \times N_\partial}.
\end{aligned}$$

Matrix form:

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\end{aligned}$$

The underlying Stokes-Dirac structure is preserved as a Dirac structure: [Egger et al., 2018]

$$\underline{e_q}^\top M_q \frac{d}{dt}\underline{\alpha_q} + \underline{e_p}^\top M_p \frac{d}{dt}\underline{\alpha_p} = \underline{y_\partial}^\top M_\partial \underline{u_\partial}$$

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Proof: $\underline{e_q}^\top (D\,\underline{e_p} + B\,\underline{u_\partial}) + \underline{e_p}^\top (-D^\top \mathbf{e_q}) = \underline{e_q}^\top B\,\underline{u_\partial} = \underline{y_\partial}^\top M_\partial\, \underline{u_\partial}$

Open-Loop Discretization 4/5 - Constitutive relations

Constitutive equations:

$$\boldsymbol{e}_q = \overline{\overline{T}}\boldsymbol{\alpha}_q$$

Open-Loop Discretization 4/5 - Constitutive relations

Constitutive equations:

$$\boldsymbol{e}_q = \overline{\overline{T}}\boldsymbol{\alpha}_q \quad \xrightarrow{\text{Testing over } \boldsymbol{v}_q} \quad (\boldsymbol{e}_q, \boldsymbol{v}_q)_\Omega = (\overline{\overline{T}}\boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega$$

Open-Loop Discretization 4/5 - Constitutive relations

Constitutive equations:

$$e_q = \overline{\overline{T}} \boldsymbol{\alpha}_q \quad \underset{\text{Using } \boldsymbol{\mathcal{V}}_q}{\overset{\text{Testing over } \boldsymbol{v}_q}{\Longrightarrow}} \quad \begin{aligned} (e_q, \boldsymbol{v}_q)_\Omega &= (\overline{\overline{T}} \boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega \\ \underbrace{\left[\int_\Omega \vec{\boldsymbol{\Phi}}_q \cdot \vec{\boldsymbol{\Phi}}_q^\top \right]}_{M_q} \underline{e}_q &= \underbrace{\left[\int_\Omega \vec{\boldsymbol{\Phi}}_q \cdot \overline{\overline{T}} \, \vec{\boldsymbol{\Phi}}_q^\top \right]}_{M_{\overline{\overline{T}}}} \underline{\alpha}_q \end{aligned}$$

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$$\boxed{M_q\, \underline{e_q} = M_{\overline{\overline{T}}}\, \underline{\alpha_q}, \qquad M_p \underline{e_p} = M_{\frac{1}{\rho}}\, \underline{\alpha_p}}$$

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Port-Hamiltonian Differential-Algebraic Equation (PHDAE): [Beattie et al., 2018]

$$(PHDAE) \begin{cases} \begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix} \dfrac{d}{dt} \begin{bmatrix} \underline{\alpha_q} \\ \underline{\alpha_p} \end{bmatrix} = \begin{bmatrix} 0 & D \\ -D^\top & 0 \end{bmatrix} \begin{bmatrix} \underline{e_q} \\ \underline{e_p} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \underline{u_\partial}, \\ \begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix} \begin{bmatrix} \underline{e_q} \\ \underline{e_p} \end{bmatrix} = \begin{bmatrix} M_{\overline{\overline{T}}} & 0 \\ 0 & M_{\frac{1}{\rho}} \end{bmatrix} \begin{bmatrix} \underline{\alpha_q} \\ \underline{\alpha_p} \end{bmatrix} \\ M_\partial \underline{y_\partial} = B^\top \underline{e_q}. \end{cases}$$

Discrete Hamiltonian:

$$
\begin{aligned}
\mathcal{H}_d(t) &:= \mathcal{H}\left(\boldsymbol{\alpha}_q^d(t), \alpha_p^d(t)\right) \\
&= \frac{1}{2}\int_\Omega \boldsymbol{\alpha}_q^d \cdot \overline{\overline{T}} \cdot \boldsymbol{\alpha}_q^d + \alpha_p^d \frac{1}{\rho} \alpha_p^d \\
&= \frac{1}{2}\underline{\alpha}_q^\top \underbrace{\left[\int_\Omega \vec{\boldsymbol{\Phi}}_q \cdot \overline{\overline{T}} \cdot \vec{\boldsymbol{\Phi}}_q^\top\right]}_{M_{\overline{\overline{T}}}} \underline{\alpha}_q + \frac{1}{2}\underline{\alpha}_p^\top \underbrace{\left[\int_\Omega \Phi_p \cdot \frac{1}{\rho}\, \Phi_p^\top\right]}_{M_{\frac{1}{\rho}}} \underline{\alpha}_p \\
&= \frac{1}{2}\underline{\alpha}_q^\top M_{\overline{\overline{T}}}\, \underline{\alpha}_q + \frac{1}{2}\underline{\alpha}_p^\top M_{\frac{1}{\rho}}\, \underline{\alpha}_p
\end{aligned}
$$

Open-Loop Discretization 5/5 - Discrete Hamiltonian

Discrete Hamiltonian:

$$
\begin{aligned}
\mathcal{H}_d(t) &:= \mathcal{H}\left(\boldsymbol{\alpha}_q^d(t), \alpha_p^d(t)\right) \\
&= \frac{1}{2}\int_\Omega \boldsymbol{\alpha}_q^d \cdot \overline{\overline{T}} \cdot \boldsymbol{\alpha}_q^d + \alpha_p^d \frac{1}{\rho} \alpha_p^d \\
&= \frac{1}{2}\underline{\alpha_q}^\top \underbrace{\left[\int_\Omega \vec{\boldsymbol{\Phi}}_q \cdot \overline{\overline{T}} \cdot \vec{\boldsymbol{\Phi}}_q^\top\right]}_{M_{\overline{\overline{T}}}} \underline{\alpha_q} + \frac{1}{2}\underline{\alpha_p}^\top \underbrace{\left[\int_\Omega \Phi_p \cdot \frac{1}{\rho}\, \Phi_p^\top\right]}_{M_{\frac{1}{\rho}}} \underline{\alpha_p} \\
&= \frac{1}{2}\underline{\alpha_q}^\top M_{\overline{\overline{T}}}\, \underline{\alpha_q} + \frac{1}{2}\underline{\alpha_p}^\top M_{\frac{1}{\rho}}\, \underline{\alpha_p}
\end{aligned}
$$

Discrete power balance:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H}_d(t) &= \underline{\alpha_q}^\top M_{\overline{\overline{T}}}\, \frac{d}{dt}\underline{\alpha_q} + \underline{\alpha_p}^\top M_{\frac{1}{\rho}}\, \frac{d}{dt}\underline{\alpha_p} \\
\textit{(constitutive relations)} &= \underline{e_q}^\top M_q\, \frac{d}{dt}\underline{\alpha_q} + \underline{e_p}^\top M_p\, \frac{d}{dt}\underline{\alpha_p} \\
\textit{(Dirac structure)} &= \underline{y_\partial}^\top M_\partial\, \underline{u_\partial} \\
&:= \left\langle \underline{u_\partial}, \underline{y_\partial} \right\rangle_\partial,
\end{aligned}
$$

Closed-Loop

Next step: Close the loop (IBC)

$$Z\, \boldsymbol{e_q} \cdot \mathbf{n} + e_p \big|_{\partial\Omega} = 0$$

Closed-Loop

Next step: Close the loop (IBC)

$$Z\,\mathbf{e}_q \cdot \mathbf{n} + e_p\big|_{\partial\Omega} = 0 \quad \overset{\text{in/out-put}}{\Longrightarrow} \quad u_\partial = -Z y_\partial$$

Closed-Loop

Next step: Close the loop (IBC)

$$
Z\, \mathbf{e}_q \cdot \mathbf{n} + e_p \big|_{\partial\Omega} = 0 \quad \overset{\text{in/out-put}}{\Longrightarrow} \quad u_\partial = -Z y_\partial
$$

$$
\overset{\text{similarly}}{\Longrightarrow} \quad \underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top \right]}_{M_\partial} \underline{u}_\partial = - \underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot Z\, \Psi_\partial^\top \right]}_{M_z} \underline{y}_\partial,
$$

Closed-Loop

Next step: Close the loop (IBC)

$$Z\, e_q \cdot \mathbf{n} + e_p \big|_{\partial\Omega} = 0 \quad \overset{\text{in/out-put}}{\Longrightarrow} \quad u_\partial = -Z y_\partial$$

$$\overset{\text{similarly}}{\Longrightarrow} \quad \underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top\right]}_{M_\partial} \underline{u}_\partial = -\underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot Z\, \Psi_\partial^\top\right]}_{M_z} \underline{y}_\partial,$$

$$\begin{cases} M_\partial \underline{u}_\partial = -M_z \underline{y}_\partial \\ M_\partial \underline{y}_\partial = B^\top \underline{e}_q \end{cases} \quad \Longrightarrow \quad \underline{u}_\partial = -M_\partial^{-1} M_z M_\partial^{-1} B^\top \underline{e}_q$$

Closed-Loop

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$$Z\, \boldsymbol{e}_q \cdot \mathbf{n} + e_p \big|_{\partial\Omega} = 0 \quad \overset{\text{in/out-put}}{\Longrightarrow} \quad u_\partial = -Z y_\partial$$

$$\overset{\text{similarly}}{\Longrightarrow} \quad \underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top\right]}_{M_\partial} \underline{u_\partial} = -\underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot Z\, \Psi_\partial^\top\right]}_{M_z} \underline{y_\partial},$$

$$\begin{cases} M_\partial \underline{u_\partial} = -M_z \underline{y_\partial} \\ M_\partial \underline{y_\partial} = B^\top \underline{e_q} \end{cases} \quad \Longrightarrow \quad \underline{u_\partial} = -M_\partial^{-1}\, M_z\, M_\partial^{-1}\, B^\top\, \underline{e_q}$$

Substitution of the control term in the system:

$$+B\,\underline{u_\partial} = -B\, M_\partial^{-1}\, M_z\, M_\partial^{-1}\, B^\top\, \underline{e_q},$$

Closed-Loop

Next step: Close the loop (IBC)

$$Z\, \mathbf{e}_q \cdot \mathbf{n} + e_p\big|_{\partial\Omega} = 0 \quad \overset{\text{in/out-put}}{\Longrightarrow} \quad u_\partial = -Z y_\partial$$

$$\overset{\text{similarly}}{\Longrightarrow} \quad \underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top\right]}_{M_\partial} \underline{u}_\partial = -\underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot Z\, \Psi_\partial^\top\right]}_{M_z} \underline{y}_\partial,$$

$$\begin{cases} M_\partial \underline{u}_\partial = -M_z \underline{y}_\partial \\ M_\partial \underline{y}_\partial = B^\top \underline{e}_q \end{cases} \quad \Longrightarrow \quad \underline{u}_\partial = -M_\partial^{-1} M_z M_\partial^{-1} B^\top \underline{e}_q$$

Substitution of the control term in the system:

$$+B\, \underline{u}_\partial = -\underbrace{B\, M_\partial^{-1} M_z M_\partial^{-1} B^\top}_{R_z} \underline{e}_q, \qquad \text{with rank}(R_z) \leq N_\partial$$

Closed-Loop

Next step: Close the loop (IBC)

$$Z\, e_q \cdot \mathbf{n} + e_p\big|_{\partial\Omega} = 0 \quad \overset{\text{in/out-put}}{\Longrightarrow} \quad u_\partial = -Z y_\partial$$

$$\overset{\text{similarly}}{\Longrightarrow} \quad \underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top\right]}_{M_\partial} \underline{u}_\partial = -\underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot Z\, \Psi_\partial^\top\right]}_{M_z} \underline{y}_\partial,$$

$$\begin{cases} M_\partial \underline{u}_\partial = -M_z \underline{y}_\partial \\ M_\partial \underline{y}_\partial = B^\top\, \underline{e}_q \end{cases} \Longrightarrow \qquad \underline{u}_\partial = -M_\partial^{-1}\, M_z\, M_\partial^{-1}\, B^\top\, \underline{e}_q$$

Substitution of the control term in the system:

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Dissipative system: Matrix R_d appears !!

$$\underbrace{\begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix}}_{M_d} \frac{d}{dt} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix} = \left(\underbrace{\begin{bmatrix} 0 & D \\ -D^\top & 0 \end{bmatrix}}_{J_d} - \underbrace{\begin{bmatrix} R_z & 0 \\ 0 & 0 \end{bmatrix}}_{R_d} \right) \begin{bmatrix} \underline{e}_q \\ \underline{e}_p \end{bmatrix},$$

Closed-Loop

Next step: Close the loop (IBC)

$$Z\, e_q \cdot \mathbf{n} + e_p\big|_{\partial\Omega} = 0 \quad \overset{\text{in/out-put}}{\Longrightarrow} \quad u_\partial = -Z y_\partial$$

$$\overset{\text{similarly}}{\Longrightarrow} \quad \underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top\right]}_{M_\partial} \underline{u}_\partial = -\underbrace{\left[\int_{\partial\Omega} \Psi_\partial \cdot Z\, \Psi_\partial^\top\right]}_{M_z} \underline{y}_\partial,$$

$$\begin{cases} M_\partial \underline{u}_\partial = -M_z \underline{y}_\partial \\ M_\partial \underline{y}_\partial = B^\top \underline{e}_q \end{cases} \quad \Longrightarrow \quad \underline{u}_\partial = -M_\partial^{-1}\, M_z\, M_\partial^{-1}\, B^\top\, \underline{e}_q$$

Substitution of the control term in the system:

$$+B\, \underline{u}_\partial = -\underbrace{B\, M_\partial^{-1}\, M_z\, M_\partial^{-1}\, B^\top}_{R_z} \underline{e}_q, \qquad \text{with } \operatorname{rank}(R_z) \leq N_\partial$$

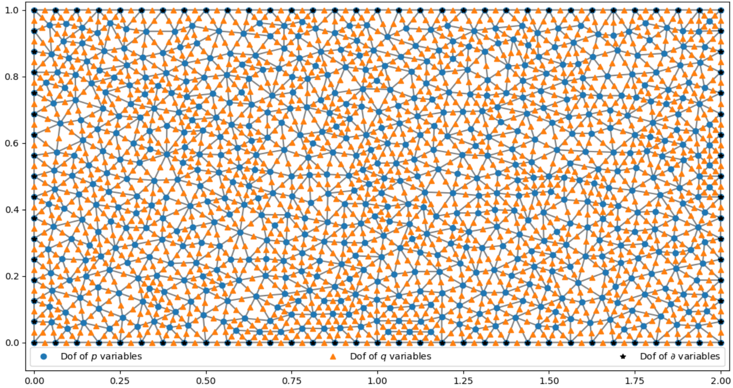
Dissipative system: Matrix R_d appears !!

$$\underbrace{\begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix}}_{M_d} \frac{d}{dt} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix} = \left(\underbrace{\begin{bmatrix} 0 & D \\ -D^\top & 0 \end{bmatrix}}_{J_d} - \underbrace{\begin{bmatrix} R_z & 0 \\ 0 & 0 \end{bmatrix}}_{R_d} \right) \begin{bmatrix} \underline{e}_q \\ \underline{e}_p \end{bmatrix},$$

$$\frac{d}{dt} \mathcal{H}_d(t) = \langle \underline{u}_\partial, \underline{y}_\partial \rangle_\partial = -\underline{y}_\partial^{\ \top} M_z \underline{y}_\partial \leq 0.$$

Mesh and finite elements

$$\Omega = (0, L_x) \times (0, L_y), \qquad L_x = 2,\ L_y = 1$$
$$\boldsymbol{\mathcal{V}}_q := \mathrm{RT}_0, \quad \mathcal{V}_p := \mathcal{P}_1 \quad \text{and} \quad \mathcal{V}_\partial := \mathrm{tr}(\mathcal{P}_1).$$
$$N_q := 1998, \quad N_p := 699 \quad \text{and} \quad N_\partial := 96.$$



Mesh and finite elements

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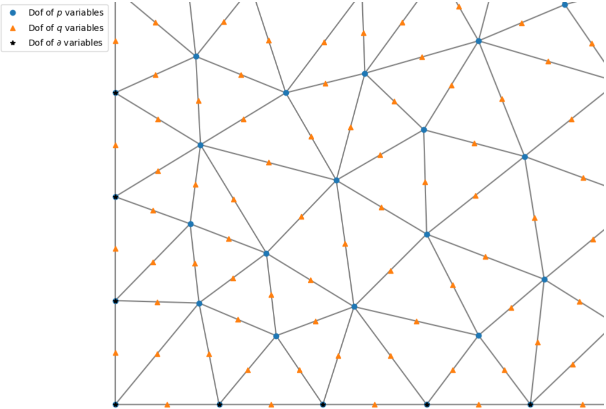
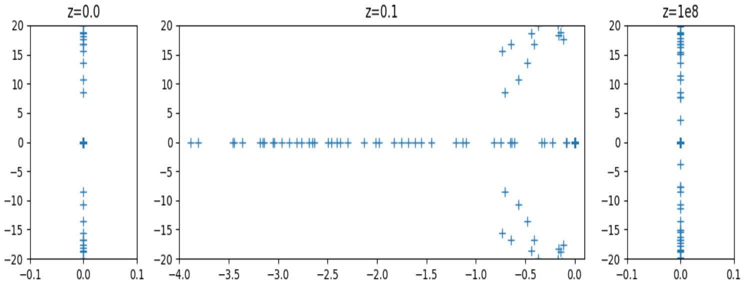


Spectrum: Generalized eigenvalue problem

$$(J_d - R_d)\, Q_d\, \mathbf{v} = \lambda M_d\, \mathbf{v}$$

$$\lambda = \beta + i\omega$$



- $z = 0 \implies \lambda = i\omega$ (Dirichlet BC), corresponds to $\mathcal{J}$ skew-adjoint.
- $z = \infty \approx 10^8 \implies \lambda = i\omega$ (Neumann BC), corresponds to $\mathcal{J}$ skew-adjoint.
- $z = 0.1 \implies \lambda = \beta + i\omega$ (IBC), corresponds to $\mathcal{J}$ *not* skew-adjoint, $\beta < 0$.

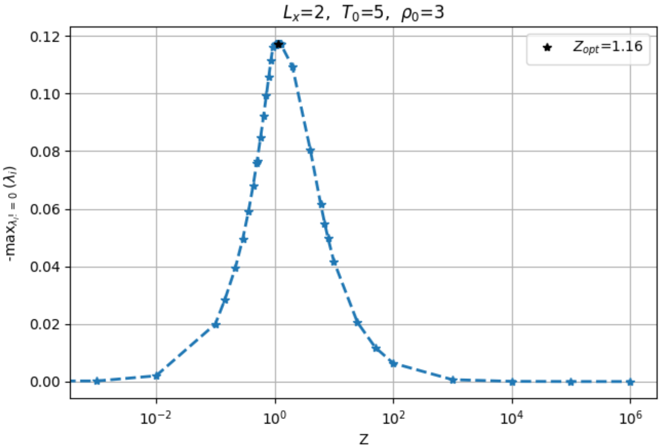
Spectral Analysis 2/3

Spectrum

Spectral Analysis 3/3

Decay rate

$$\lambda_n = \lambda_n(Z), \qquad -\max_{\lambda_n \neq 0} \beta_n$$



Simulation

Space parameters:

- $\rho(x) = x^2\,(2-x)+1$
- $\overline{\overline{T}}(x,y) = \begin{bmatrix} x^2+1 & y \\ y & x+1 \end{bmatrix}$
- $Z\big|_{\Gamma_1\cap\Gamma_3} = 1$, $Z\big|_{\Gamma_2\cap\Gamma_4} = 0.5$

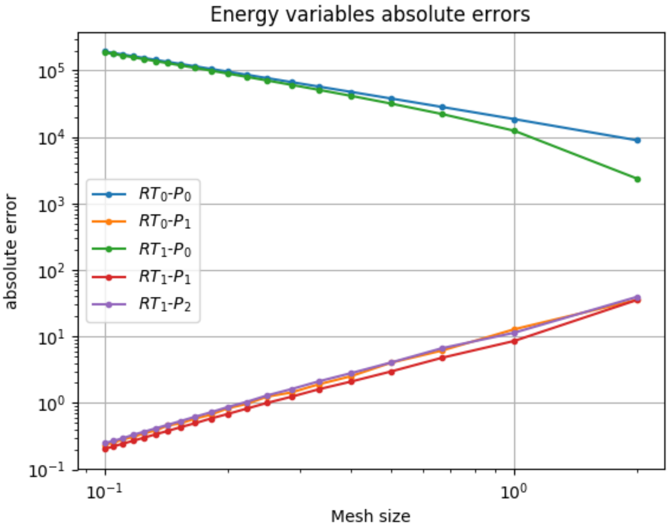
Time integration:

- Crank-Nicolson
- $t_f = 5$
- $\Delta t = 10^{-3}$

Simulation

Convergence - Errors with known analytical solution

<u>*Absolute error of energy variables:*</u> $\sup_{0\leq t\leq t_f} \|\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}_{\text{exact}}(t)\|_{\mathcal{H}}$



Conclusion and perspectives

Conclusion

- Structure preserving discretization of the wave equation with boundary damping.
- In continuous model, the dissipation was hidden in the domain of the operator.
- At discrete level, the dissipation appeared explicitly in a matrix R_d thanks to PFEM. R_d is with low rank, since it accounts for boundary damping.
- Simulation of different examples: Anistropic & Heterogeneous.
- Numerical evidence of convergence.

Perspectives

- Finite element convergence analysis and error estimation [Serhani et al., 2019a] .
- Structure-preserving discretization of heat problem (PHDAE) [Serhani et al., 2019b] , [Serhani et al., 2019c] .
- Structure-preserving model reduction.
- Coupled problems from thermoelasticity.
- Discretization of acoustically time-varying impedance [Monteghetti et al., 2018] .

THANK YOU !

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Different representations

Infinite-dimensional Hamiltonian system:

$$\partial_t \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix} = \begin{bmatrix} 0 & \textbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix}, \qquad \boldsymbol{e}_q = \overline{\overline{T}} \boldsymbol{\alpha}_q, \; e_p = \frac{1}{\rho} \alpha_p$$

Different representations

Infinite-dimensional Hamiltonian system:

$$\partial_t \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix}, \qquad \boldsymbol{e}_q = \overline{\overline{T}} \boldsymbol{\alpha}_q, \; e_p = \frac{1}{\rho} \alpha_p$$

Energy representation:

$$\partial_t \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \overline{\overline{T}} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix},$$

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Energy representation:

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Co-energy representation:

$$\partial_t \begin{bmatrix} \overline{\overline{T}}^{-1} & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix}$$

Different representations

Infinite-dimensional Hamiltonian system:

$$\partial_t \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix}, \qquad \boldsymbol{e}_q = \overline{\overline{T}} \boldsymbol{\alpha}_q, \; e_p = \frac{1}{\rho} \alpha_p$$

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$$\partial_t \begin{bmatrix} \overline{\overline{T}}^{-1} & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix}$$

DISCRETE Energy representation:

$$\begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix} = \begin{bmatrix} 0 & D \\ -D^\top & 0 \end{bmatrix} \begin{bmatrix} Q_T & 0 \\ 0 & Q_\rho \end{bmatrix} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix}, \quad \begin{bmatrix} Q_T & 0 \\ 0 & Q_\rho \end{bmatrix} = \begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix}^{-1} \begin{bmatrix} M_{\overline{\overline{T}}} & 0 \\ 0 & M_\rho \end{bmatrix}$$

Different representations

Infinite-dimensional Hamiltonian system:

$$\partial_t \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix}, \qquad \boldsymbol{e}_q = \overline{\overline{T}} \boldsymbol{\alpha}_q, \; e_p = \frac{1}{\rho} \alpha_p$$

Energy representation:

$$\partial_t \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \overline{\overline{T}} & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_q \\ \alpha_p \end{bmatrix},$$

Co-energy representation:

$$\partial_t \begin{bmatrix} \overline{\overline{T}}^{-1} & 0 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{grad} \\ \mathrm{div} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \end{bmatrix}$$

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$$\begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix} = \begin{bmatrix} 0 & D \\ -D^\top & 0 \end{bmatrix} \begin{bmatrix} Q_T & 0 \\ 0 & Q_\rho \end{bmatrix} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix}, \quad \begin{bmatrix} Q_T & 0 \\ 0 & Q_\rho \end{bmatrix} = \begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix}^{-1} \begin{bmatrix} M_{\overline{\overline{T}}} & 0 \\ 0 & M_\rho \end{bmatrix}$$

DISCRETE Co-energy representation:

$$\begin{bmatrix} M_{\overline{\overline{T}}} & 0 \\ 0 & M_\rho \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \underline{e}_q \\ \underline{e}_p \end{bmatrix} = \begin{bmatrix} 0 & D \\ -D^\top & 0 \end{bmatrix} \begin{bmatrix} \underline{e}_q \\ \underline{e}_p \end{bmatrix}$$

Finite element families optimal choice

Following [Serhani et al., 2019] ,

$$\mathcal{V}_q \times \mathcal{V}_p \times \mathcal{V}_\partial = \mathrm{RT}_k(\Omega) \times \mathcal{P}_l(\Omega) \times \mathcal{P}_m(\partial\Omega)$$

Hamiltonian error estimation

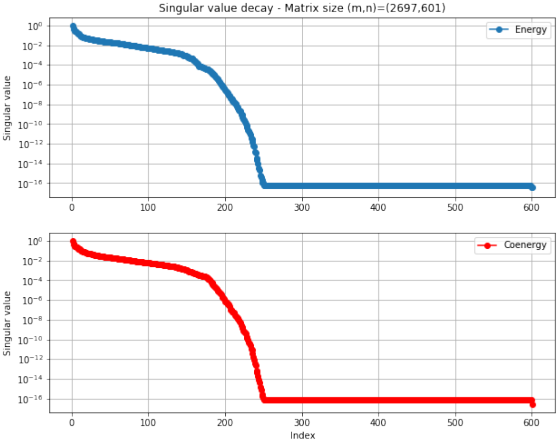
$$\left|\mathcal{H} - \mathcal{H}_d\right| = o(h^s)$$

Energy variables error estimation

$$\left\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_d\right\|_{\mathcal{H}} = o(h^r)$$

Model reduction: Structure-preserving Proper Orthogonal Decomposition (POD) reduction

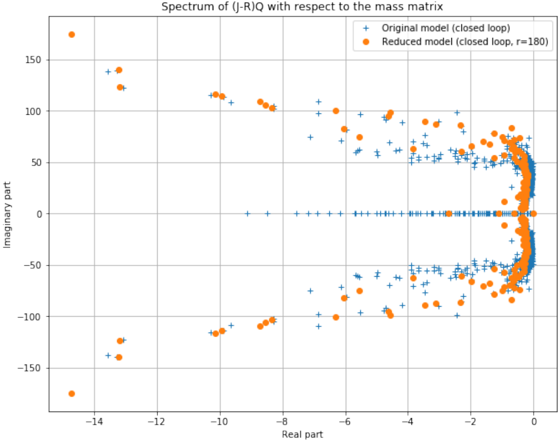
Figure: SVD on snapshots of the energy and co-energy variables



Chaturantabut, S., Beattie, C., and Gugercin, S. (2016). Structure-preserving model reduction for nonlinear Port-Hamiltonian systems. *SIAM Journal on Scientific Computing*, 38(5), B837–B865

Model reduction: Structure-preserving Proper Orthogonal Decomposition (POD) reduction

Figure: Spectrum, tolerance $\varepsilon = 10^{-4}$



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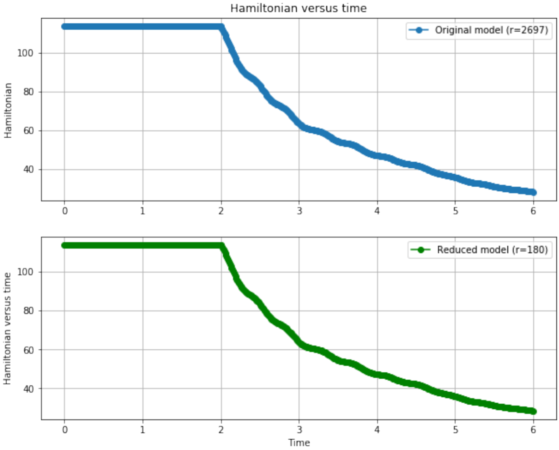
Model reduction: Structure-preserving Proper Orthogonal Decomposition (POD) reduction

Figure: Hamiltonian, tolerance $\varepsilon = 10^{-4}$



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Weighted scalar product & discrete Hamiltonian

New scalar products:

$$\left\langle \mathbf{v}_1, \mathbf{v}_2 \right\rangle_q := \mathbf{v}_1^\top M_q \, \mathbf{v}_2, \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{N_q},$$

$$\left\langle \mathbf{v}_1, \mathbf{v}_2 \right\rangle_p := \mathbf{v}_1^\top M_p \, \mathbf{v}_2, \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{N_p},$$

$$\left\langle \mathbf{v}_1, \mathbf{v}_2 \right\rangle_\partial := \mathbf{v}_1^\top M_\partial \, \mathbf{v}_2, \quad \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{N_\partial},$$

$Q_T, Q_\rho \overset{sym}{\geq} 0$. Also y_∂ is exactly the conjugated output of u_∂ with respect to $\langle \cdot, \cdot \rangle_\partial$.

Discrete Hamiltonian:

$$\begin{aligned} \mathcal{H}_d(t) &:= \frac{1}{2} \int_\Omega \boldsymbol{\alpha}_q^d \cdot \overline{\overline{T}} \cdot \boldsymbol{\alpha}_q^d + \alpha_p^d \frac{1}{\rho} \alpha_p^d \\ &= \frac{1}{2} \langle \underline{\alpha}_q, Q_T \, \underline{\alpha}_q \rangle_q + \frac{1}{2} \langle \underline{\alpha}_p, Q_\rho \, \underline{\alpha}_p \rangle_p . \end{aligned}$$

Discrete power balance:

$$\frac{d}{dt} \mathcal{H}_d(t) = \underline{y}_\partial^\top M_\partial \underline{u}_\partial := \langle \underline{u}_\partial, \underline{y}_\partial \rangle_\partial,$$