Structure-Preserving Finite Volume Method for 2D Linear and Non-Linear Port-Hamiltonian Systems

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Valparaíso

Overview

Linear wave equation

2D wave

Consider the linear wave equation on the two-dimensional domain $\Omega = (0, l_x) \times (0, l_y)$,

$$\rho(\mathbf{x}) \frac{\partial^2}{\partial t^2} u(\mathbf{x}, t) = \mathrm{div} \left(T(\mathbf{x}) \nabla u(\mathbf{x}, t) \right)$$

where $\rho(\mathbf{x})$ and $T(\mathbf{x})$ are the mass density and Young's modulus.

Linear wave equation

2D wave

Consider the linear wave equation on the two-dimensional domain $\Omega = (0, l_x) \times (0, l_y)$,

$$\rho(x,y)\frac{\partial^2}{\partial t^2}u(x,y,t) = \frac{\partial}{\partial x}\left(T(x,y)\frac{\partial}{\partial x}u(x,y,t)\right) + \frac{\partial}{\partial y}\left(T(x,y)\frac{\partial}{\partial y}u(x,y,t)\right) \tag{1}$$

where $\rho(x,y)$ and $T(x,y)$ are the mass density and Young's modulus.

Linear wave equation

2D wave equation

$$\rho \frac{\partial^2}{\partial t^2} u = \frac{\partial}{\partial x}\left(T \frac{\partial}{\partial x} u\right) + \frac{\partial}{\partial y}\left(T \frac{\partial}{\partial y} u\right)$$

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Associated Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int_\Omega \rho \left(\frac{\partial}{\partial t} u\right)^2 + T\left(\frac{\partial}{\partial x} u\right)^2 + T\left(\frac{\partial}{\partial y} u\right)^2$$

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strain: $q^1 = \frac{\partial}{\partial x} u$, $q^2 = \frac{\partial}{\partial y} u$ and *momentum*: $p = \rho \frac{\partial}{\partial t} u$

(3)

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The wave equation can be written as follows

$$\underbrace{\frac{\partial}{\partial t}\begin{bmatrix} q^1 \\ q^2 \\ p \end{bmatrix}}_{f:\ \text{flow}} = \underbrace{\begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{bmatrix}}_{\mathcal{J} = -\mathcal{J}^*} \underbrace{\begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & \frac{1}{\rho} \end{bmatrix}\begin{bmatrix} q^1 \\ q^2 \\ p \end{bmatrix}}_{e:\ \text{effort}} \tag{1}$$

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By the *flow-effort* notation we get the port-Hamiltonian system,

$$(pHs)\begin{cases} f = \mathcal{J} e \\ \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \text{related to the trace } \mathrm{tr}(e) \end{cases} \quad (2)$$

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Discretization Methods

Finite Element / Galerkin Approachs

- Mixed finite element method [Golo et al., 2004]
- Pseudo spectral finite element method [Moulla et al., 2012]
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Discrete methods based on the Geometry of the system

- Explicit simplicial discretization [Seslija et al., 2014]
- Finite volume method (FVM) [Kotyczka, 2016]
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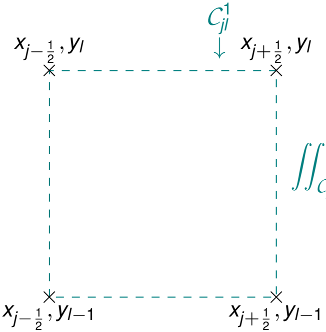
$$C^1_{jl} = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right) \times \left(y_{l-1}, y_l\right)$$



$$\iint_{C^1_{jl}} \dot{q}^1 \, dxdy = \iint_{C^1_{jl}} \frac{\partial}{\partial x} p \, dxdy$$

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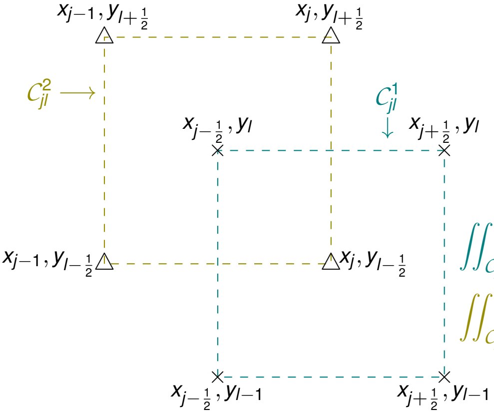

$$C_{jl}^1 = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right) \times \left(y_{l-1}, y_l\right)$$

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$$\iint_{C_{jl}^1} \dot{q}^1 \, dxdy = \iint_{C_{jl}^1} \frac{\partial}{\partial x} p \, dxdy$$

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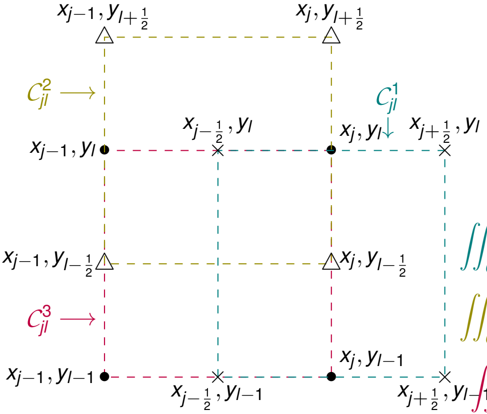

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$$\iint_{C_{jl}^1} \dot{q}^1 \, dxdy = \iint_{C_{jl}^1} \frac{\partial}{\partial x} p \, dxdy$$

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Finite volume discretization - *Natural Flux*

$$\iint_{C^1_{jl}} \dot{q}^1 \; dxdy = \iint_{C^1_{jl}} \frac{\partial}{\partial x} p \; dxdy$$

Finite volume discretization - *Natural Flux*

$$\iint_{C_{jl}^1} \dot{q}^1 \; dxdy = \int_{y_{l-1}}^{y_l} \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \frac{\partial}{\partial x} p \; dx \right) dy$$

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$$\iint_{C_{jl}^1} \dot{q}^1 \, dxdy = \int_{y_{l-1}}^{y_l} p(x_{j+\frac{1}{2}}, y) dy - \int_{y_{l-1}}^{y_l} p(x_{j-\frac{1}{2}}, y) dy$$

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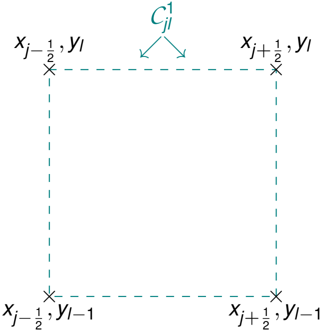

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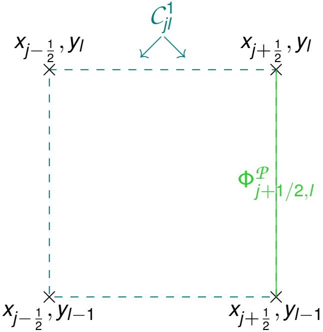

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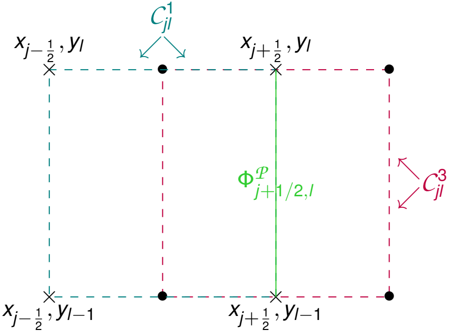

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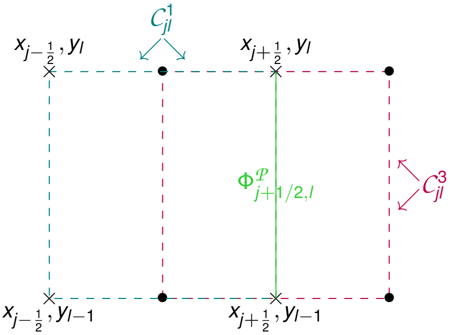

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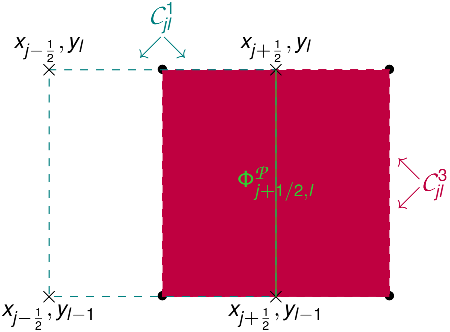

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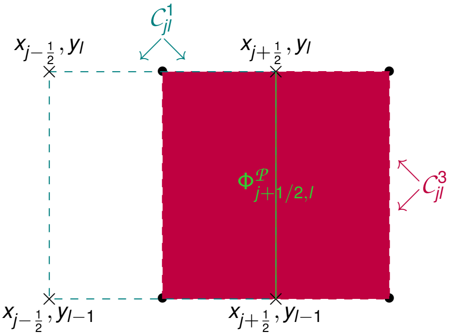

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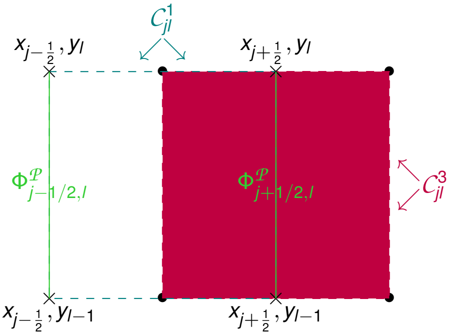
$$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \mathcal{P}_{jl}}$$

Finite volume discretization - *Natural Flux*

$$\frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} \dot{q}^1 \, dxdy = \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\big(x_{j+1/2}, y\big)\, dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\big(x_{j-1/2}, y\big)\, dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}}$$

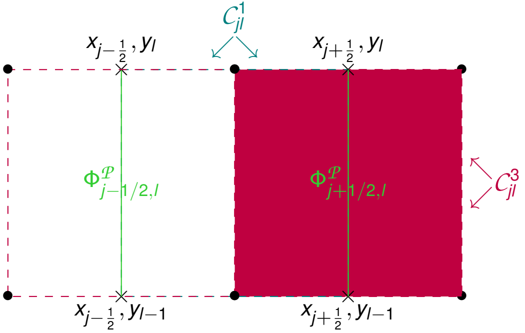


$$Q^1_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} q^1 \, dxdy$$
$$Q^2_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^2_{jl}} q^2 \, dxdy$$
$$\mathcal{P}_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^3_{jl}} p \, dxdy$$

$$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \mathcal{P}_{jl}}$$

Finite volume discretization - *Natural Flux*

$$\frac{1}{\Delta x \Delta y}\iint_{C_{jl}^1} \dot{q}^1 \, dxdy = \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j+1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j-1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}}$$

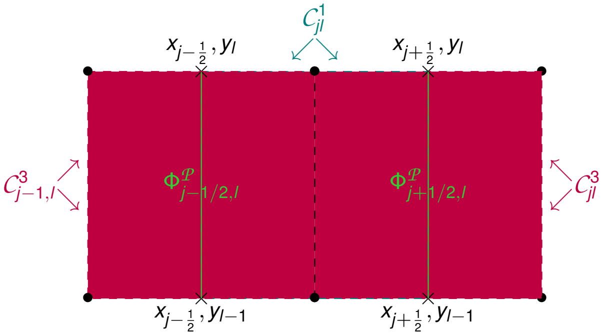


$$Q_{jl}^1 := \frac{1}{\Delta x \Delta y}\iint_{C_{jl}^1} q^1 \, dxdy$$
$$Q_{jl}^2 := \frac{1}{\Delta x \Delta y}\iint_{C_{jl}^2} q^2 \, dxdy$$
$$\mathcal{P}_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C_{jl}^3} p \, dxdy$$

$$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \mathcal{P}_{jl}}$$

Finite volume discretization - *Natural Flux*

$$\frac{1}{\Delta x \Delta y}\iint_{C_{jl}^1} \dot{q}^1 \; dxdy = \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j+1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j-1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}}$$

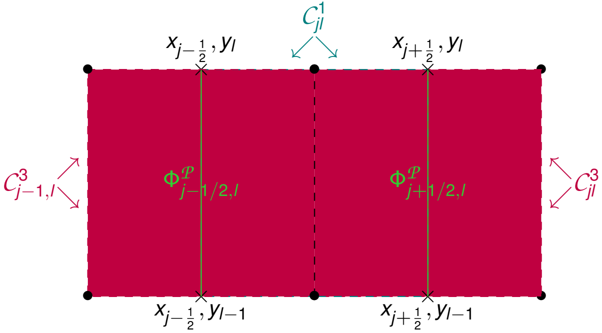


$$Q_{jl}^1 := \frac{1}{\Delta x \Delta y}\iint_{C_{jl}^1} q^1 \; dxdy$$
$$Q_{jl}^2 := \frac{1}{\Delta x \Delta y}\iint_{C_{jl}^2} q^2 \; dxdy$$
$$\mathcal{P}_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C_{jl}^3} p \; dxdy$$

$$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \mathcal{P}_{jl}}$$

Finite volume discretization - *Natural Flux*

$$\frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} \dot{q}^1 \; dxdy = \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j+1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j-1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}}$$



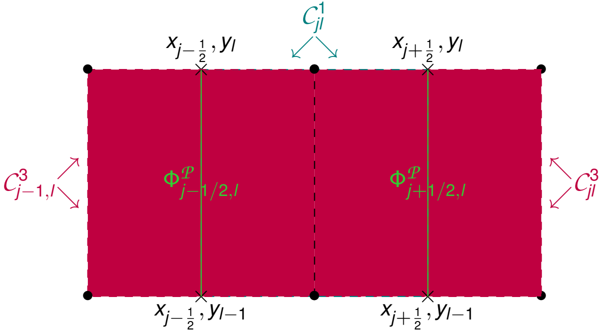
$$Q^1_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} q^1 \; dxdy$$
$$Q^2_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^2_{jl}} q^2 \; dxdy$$
$$\mathcal{P}_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^3_{jl}} p \; dxdy$$

$$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \mathcal{P}_{jl}}$$

$$\boxed{\Phi^{\mathcal{P}}_{j-1/2,l} \approx \mathcal{P}_{j-1,l}}$$

Finite volume discretization - *Natural Flux*

$$\frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} \dot{q}^1 \, dxdy = \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\big(x_{j+1/2},y\big)\,dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\big(x_{j-1/2},y\big)\,dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}}$$



$$Q^1_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} q^1 \, dxdy$$
$$Q^2_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^2_{jl}} q^2 \, dxdy$$
$$\mathcal{P}_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^3_{jl}} p \, dxdy$$

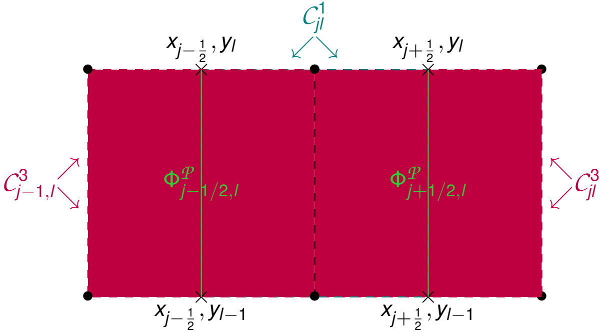
$$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \mathcal{P}_{jl}}$$

$$\boxed{\Phi^{\mathcal{P}}_{j-1/2,l} \approx \mathcal{P}_{j-1,l}}$$

$$\dot{Q}^1_{jl} = \frac{1}{\Delta x}\Big(\mathcal{P}_{j,l} - \mathcal{P}_{j-1,l}\Big).$$

Finite volume discretization - *Natural Flux*

$$\frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} \dot{q}^1 \, dxdy = \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j+1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j-1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}}$$



$$Q^1_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} q^1 \, dxdy$$

$$Q^2_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^2_{jl}} q^2 \, dxdy$$

$$\mathcal{P}_{jl} := \frac{1}{\Delta x \Delta y}\iint_{C^3_{jl}} p \, dxdy$$

$$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \mathcal{P}_{jl}}$$

$$\boxed{\Phi^{\mathcal{P}}_{j-1/2,l} \approx \mathcal{P}_{j-1,l}}$$

$$\dot{Q}^1_{jl} = \frac{1}{\Delta x}\left(\mathcal{P}_{j,l} - \mathcal{P}_{j-1,l}\right).$$

Similarly we get $\dot{Q}^2_{jl} = \frac{1}{\Delta y}\left(\mathcal{P}_{j,l} - \mathcal{P}_{j,l-1}\right)$ and $\dot{\mathcal{P}}_{jl} = \frac{1}{\Delta x}\left(Q^1_{j+1,l} - Q^1_{j,l}\right) + \frac{1}{\Delta y}\left(Q^2_{j,l+1} - Q^2_{j,l}\right)$

Natural Flux - matrix form

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Natural Flux - matrix form

$$\begin{cases} \underbrace{\begin{bmatrix} \dot{Q}^1 \\ \dot{Q}^2 \\ \dot{\mathcal{P}} \end{bmatrix}}_{F^d} = \underbrace{\begin{bmatrix} 0 & 0 & D_1 \\ 0 & 0 & \tilde{D}_1 \\ -D_1^T & -\tilde{D}_1^T & 0 \end{bmatrix}}_{J_1^d = -J_1^d} \underbrace{\begin{bmatrix} Q^1 \\ Q^2 \\ \mathcal{P} \end{bmatrix}}_{E^d} + \underbrace{\begin{bmatrix} 0 & 0 \\ g_1 & \tilde{g}_1 \end{bmatrix}}_{g_1^d} \underbrace{\begin{bmatrix} Q^1_{\partial_x} \\ Q^2_{\partial_y} \end{bmatrix}}_{E^d_\partial}, \\ \underbrace{\begin{bmatrix} \mathcal{P}_{\partial_x} \\ \mathcal{P}_{\partial_y} \end{bmatrix}}_{F^d_\partial} = \underbrace{\begin{bmatrix} 0 & 0 \\ g_1 & \tilde{g}_1 \end{bmatrix}^T}_{{g_1^d}^T} \underbrace{\begin{bmatrix} Q^1 \\ Q^2 \\ \mathcal{P} \end{bmatrix}}_{E^d}, \end{cases}$$

Natural Flux - matrix form

$$
\begin{cases}
\underbrace{\begin{bmatrix} \dot{Q}^1 \\ \dot{Q}^2 \\ \dot{\mathcal{P}} \end{bmatrix}}_{F^d} = \underbrace{\begin{bmatrix} 0 & 0 & D_1 \\ 0 & 0 & \tilde{D}_1 \\ -D_1^T & -\tilde{D}_1^T & 0 \end{bmatrix}}_{J_1^d = -J_1^d} \underbrace{\begin{bmatrix} Q^1 \\ Q^2 \\ \mathcal{P} \end{bmatrix}}_{E^d} + \underbrace{\begin{bmatrix} 0 & 0 \\ g_1 & \tilde{g}_1 \end{bmatrix}}_{g_1^d} \underbrace{\begin{bmatrix} Q^1_{\partial_x} \\ Q^2_{\partial_y} \end{bmatrix}}_{E_\partial^d}, \\
\underbrace{\begin{bmatrix} \mathcal{P}_{\partial_x} \\ \mathcal{P}_{\partial_y} \end{bmatrix}}_{F_\partial^d} = \underbrace{\begin{bmatrix} 0 & 0 \\ g_1 & \tilde{g}_1 \end{bmatrix}^T}_{g_1^{d\,T}} \underbrace{\begin{bmatrix} Q^1 \\ Q^2 \\ \mathcal{P} \end{bmatrix}}_{E^d},
\end{cases}
$$

where,

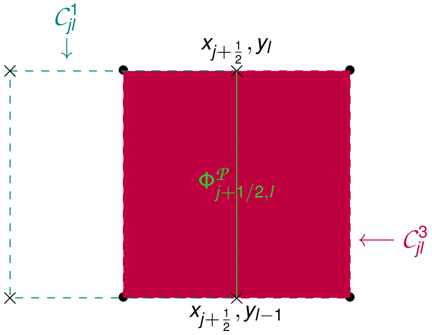
$D_1 \in \mathbb{R}^{(N_x-1)N_y \times N_x N_y}$, $\tilde{D} \in \mathbb{R}^{N_x(N_y-1) \times N_x N_y}$, $A_1 \in \mathbb{R}^{(N_y-1)N_y}$, $g_1 \in \mathbb{R}^{N_x N_y \times 2N_y}$, $\tilde{g}_1 \in \mathbb{R}^{N_x N_y \times 2N_x}$, $B \in \mathbb{R}^{N_y \times 2}$,

$$
D_1 = \frac{1}{\Delta x} \begin{bmatrix} -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad \tilde{D}_1 = \frac{1}{\Delta y} \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_1 \end{bmatrix}
$$

$$
A_1 = \begin{bmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}, \qquad g_1 = \frac{1}{\Delta x} \begin{bmatrix} -I_{N_y} & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & I_{N_y} \end{bmatrix}, \; \tilde{g}_1 = \frac{1}{\Delta y} \begin{bmatrix} B & & \\ & \ddots & \\ & & B \end{bmatrix}, \; B = \begin{bmatrix} -1 & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & 1 \end{bmatrix}.
$$

Finite volume discretization - *Leapfrog Flux*

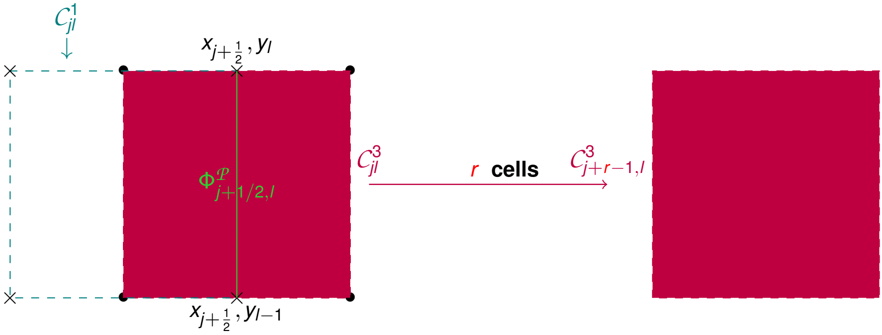
$$\frac{1}{\Delta x \Delta y}\iint_{C_{jl}^1} \dot{q}^1 \, dxdy = \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j+1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j-1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}} \tag{4}$$



(5)

Finite volume discretization - *Leapfrog Flux*

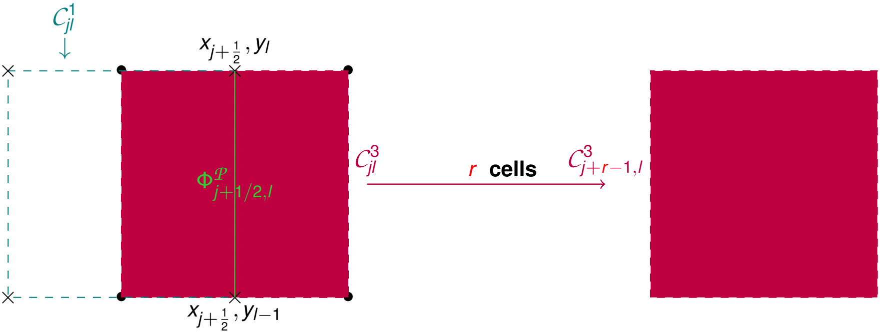
$$\frac{1}{\Delta x \Delta y}\iint_{C_{jl}^1} \dot{q}^1 \, dxdy = \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j+1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \frac{1}{\Delta x}\underbrace{\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j-1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}} \qquad (4)$$



(5)

Finite volume discretization - *Leapfrog Flux*

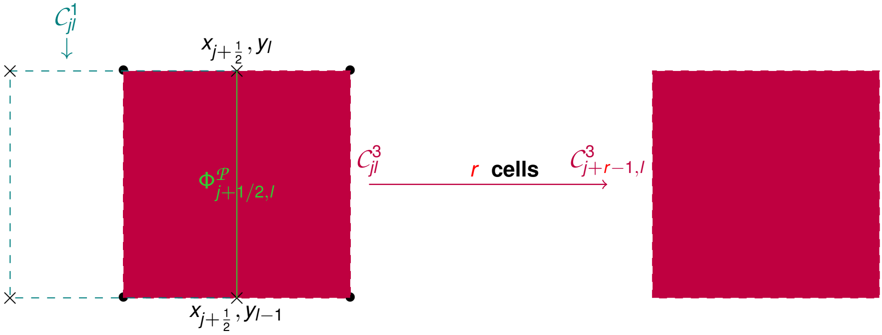
$$\frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} \dot{q}^1 \, dxdy = \underbrace{\frac{1}{\Delta x}\,\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j+1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \underbrace{\frac{1}{\Delta x}\,\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j-1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}} \tag{4}$$



$$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \Phi\left(\mathcal{P}_{j,l}, \dots, \mathcal{P}_{j+r-1,l}\right)} \quad \Phi : (v_1, \dots, v_r) \mapsto \sum_{n=1}^{r} a_{n,r} v_n \quad \text{[Fornberg and Ghrist, 1999]} \tag{5}$$

Finite volume discretization - *Leapfrog Flux*

$$\frac{1}{\Delta x \Delta y}\iint_{C^1_{jl}} \dot{q}^1 \, dxdy = \underbrace{\frac{1}{\Delta x}\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j+1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j+1/2,l}} - \underbrace{\frac{1}{\Delta x}\frac{1}{\Delta y}\int_{y_{l-1}}^{y_l} p\left(x_{j-1/2}, y\right) dy}_{\Phi^{\mathcal{P}}_{j-1/2,l}} \quad (4)$$



$\boxed{\Phi^{\mathcal{P}}_{j+1/2,l} \approx \Phi\left(\mathcal{P}_{j,l}, \dots, \mathcal{P}_{j+r-1,l}\right)}$ $\Phi : (v_1, \dots, v_r) \mapsto \sum_{n=1}^{r} a_{n,r} v_n$ [Fornberg and Ghrist, 1999]

$$\dot{Q}^1_{jl} = \frac{1}{\Delta x}\left(\Phi\left(\mathcal{P}_{j+1,l}, \dots, \mathcal{P}_{j+r,l}\right) - \Phi\left(\mathcal{P}_{j,l}, \dots, \mathcal{P}_{j-r+1,l}\right)\right) = \frac{1}{\Delta x}\sum_{n=1}^{r} a_{n,r}\left(\mathcal{P}_{j+n,l} - \mathcal{P}_{j-n+1,l}\right) \quad (5)$$

Leapfrog flux - matrix form I

$$\begin{cases} \underbrace{\begin{bmatrix} \dot{Q}^1 \\ \dot{Q}^2 \\ \dot{\mathcal{P}} \end{bmatrix}}_{F^d} = \underbrace{\begin{bmatrix} 0 & 0 & D_r \\ 0 & 0 & \tilde{D}_r \\ -D_r^T & -\tilde{D}_r^T & 0 \end{bmatrix}}_{J_r^d = -J_r^d} \underbrace{\begin{bmatrix} Q^1 \\ Q^2 \\ \mathcal{P} \end{bmatrix}}_{E^d} + \underbrace{\begin{bmatrix} 0 & 0 \\ g_r & \tilde{g}_r \end{bmatrix}}_{g_r^d} \underbrace{\begin{bmatrix} Q^1_{\partial_x} \\ Q^2_{\partial_y} \end{bmatrix}}_{E^d_\partial}, \\ \underbrace{\begin{bmatrix} \mathcal{P}_{\partial_x} \\ \mathcal{P}_{\partial_y} \end{bmatrix}}_{F^d_\partial} = \underbrace{\begin{bmatrix} 0 & 0 \\ g_r & \tilde{g}_r \end{bmatrix}^T}_{{g_r^d}^T} \underbrace{\begin{bmatrix} Q^1 \\ Q^2 \\ \mathcal{P} \end{bmatrix}}_{E^d}, \end{cases}$$

Leapfrog flux - matrix form I

$$\begin{cases} \underbrace{\begin{bmatrix} \dot{Q}^1 \\ \dot{Q}^2 \\ \dot{\mathcal{P}} \end{bmatrix}}_{F^d} = \underbrace{\begin{bmatrix} 0 & 0 & D_r \\ 0 & 0 & \tilde{D}_r \\ -D_r^T & -\tilde{D}_r^T & 0 \end{bmatrix}}_{J_r^d = -J_r^d} \underbrace{\begin{bmatrix} Q^1 \\ Q^2 \\ \mathcal{P} \end{bmatrix}}_{E^d} + \underbrace{\begin{bmatrix} 0 & 0 \\ g_r & \tilde{g}_r \end{bmatrix}}_{g_r^d} \underbrace{\begin{bmatrix} Q^1_{\partial_x} \\ Q^2_{\partial_y} \end{bmatrix}}_{E^d_\partial}, \\ \underbrace{\begin{bmatrix} \mathcal{P}_{\partial_x} \\ \mathcal{P}_{\partial_y} \end{bmatrix}}_{F^d_\partial} = \underbrace{\begin{bmatrix} 0 & 0 \\ g_r & \tilde{g}_r \end{bmatrix}^T}_{{g_r^d}^T} \underbrace{\begin{bmatrix} Q^1 \\ Q^2 \\ \mathcal{P} \end{bmatrix}}_{E^d}, \end{cases}$$

Discrete energy balance

$$\dot{\mathcal{H}}_d := \frac{1}{2}\frac{d}{dt}\left(\|Q^1\|^2 + \|Q^2\|^2 + \|\mathcal{P}\|^2\right) = \mathcal{P}_{\partial_x}^T Q^1_{\partial_x} + \mathcal{P}_{\partial_y}^T Q^2_{\partial_y}$$

In a compact form

$$(F^d)^T E^d = (F^d_\partial)^T E^d_\partial$$

Leapfrog flux - matrix form II

$D_r \in \mathbb{R}^{(N_x-1)N_y \times N_x N_y}$, $\tilde{D}_r \in \mathbb{R}^{N_x (N_y-1)\times N_x N_y}$, $A_r \in \mathbb{R}^{(N_y-1)N_y}$, $g_r \in \mathbb{R}^{N_x N_y \times 2N_y}$, $B_n \in \mathbb{R}^{N_y \times N_y}$, $\tilde{g}_r \in \mathbb{R}^{N_x N_y \times 2N_x}$, $C_r \in \mathbb{R}^{N_y \times 2}$,

$$D_r = \frac{1}{\Delta x}\begin{bmatrix} -a_{1,r} & 0 & \cdots & 0 & a_{1,r} & \cdots & a_{r,r} & & \\ \vdots & \ddots & \ddots & & \ddots & \ddots & & \ddots & \\ -a_{r,r} & & \ddots & \ddots & & \ddots & \ddots & & a_{r,r} \\ & \ddots & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & -a_{r,r} & \cdots & -a_{1,r} & 0 & \cdots & 0 & a_{1,r} \end{bmatrix}, \quad \tilde{D}_r = \frac{1}{\Delta y}\begin{bmatrix} A_r & & \\ & \ddots & \\ & & A_r \end{bmatrix},$$

$$A_r = \begin{bmatrix} -a_{1,r} & a_{1,r} & \cdots & a_{r,r} & & & \\ \vdots & \ddots & \ddots & & \ddots & & \\ -a_{r,r} & & \ddots & \ddots & & & a_{r,r} \\ & \ddots & & \ddots & \ddots & & \vdots \\ & & -a_{r,r} & \cdots & -a_{1,r} & & a_{1,r} \end{bmatrix}, \quad g_r = \frac{1}{\Delta x}\begin{bmatrix} -B & 0 \\ 0 & \mathrm{flip}(B) \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix}, \quad B_n = b_n I_{N_y}, \quad \tilde{g}_r = \frac{1}{\Delta y}\begin{bmatrix} C_r & & \\ & \ddots & \\ & & C_r \end{bmatrix}, \quad C_r = \begin{bmatrix} -b_1 & 0 \\ \vdots & \vdots \\ -b_r & 0 \\ 0 & b_r \\ \vdots & \vdots \\ 0 & b_1 \end{bmatrix}$$

where $b_n = \sum_{i=n}^{r} a_{r,i}$. and $\mathrm{flip}(B) := \left(B_r, \ldots, B_1\right)^T$.

Leapfrog flux - matrix form II

$D_r \in \mathbb{R}^{(N_x-1)N_y \times N_x N_y}$, $\tilde{D}_r \in \mathbb{R}^{N_x(N_y-1)\times N_x N_y}$, $A_r \in \mathbb{R}^{(N_y-1)N_y}$, $g_r \in \mathbb{R}^{N_x N_y \times 2N_y}$, $B_n \in \mathbb{R}^{N_y \times N_y}$, $\tilde{g}_r \in \mathbb{R}^{N_x N_y \times 2N_x}$, $C_r \in \mathbb{R}^{N_y \times 2}$,

$$D_r = \frac{1}{\Delta x}\begin{bmatrix} -a_{1,r} & 0 & \cdots & 0 & a_{1,r} & \cdots & a_{r,r} & & \\ \vdots & \ddots & \ddots & & \ddots & \ddots & & \ddots & \\ -a_{r,r} & & \ddots & \ddots & & \ddots & \ddots & & a_{r,r} \\ & \ddots & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & -a_{r,r} & \cdots & -a_{1,r} & 0 & \cdots & 0 & a_{1,r} \end{bmatrix}, \quad \tilde{D}_r = \frac{1}{\Delta y}\begin{bmatrix} A_r & & \\ & \ddots & \\ & & A_r \end{bmatrix},$$

$$A_r = \begin{bmatrix} -a_{1,r} & a_{1,r} & \cdots & a_{r,r} & & \\ \vdots & \ddots & \ddots & & \ddots & \\ -a_{r,r} & & \ddots & \ddots & & a_{r,r} \\ & \ddots & & \ddots & \ddots & \vdots \\ & & -a_{r,r} & \cdots & -a_{1,r} & a_{1,r} \end{bmatrix}, \quad g_r = \frac{1}{\Delta x}\begin{bmatrix} -B & 0 \\ 0 & \text{flip}(B) \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix}, \quad B_n = b_n I_{N_y}, \quad \tilde{g}_r = \frac{1}{\Delta y}\begin{bmatrix} C_r & & \\ & \ddots & \\ & & C_r \end{bmatrix}, \quad C_r = \begin{bmatrix} -b_1 & 0 \\ \vdots & \vdots \\ -b_r & 0 \\ 0 & b_r \\ \vdots & \vdots \\ 0 & b_1 \end{bmatrix}$$

where $b_n = \sum_{i=n}^{r} a_{r,i}$. and $\text{flip}(B) := \left(B_r, ..., B_1\right)^T$.

☺ We can reach higher accuracy without increasing the size of the matrices, but only by making it less sparse by increasing r.

Leapfrog flux - matrix form II

$D_r \in \mathbb{R}^{(N_x-1)N_y \times N_x N_y}$, $\tilde{D}_r \in \mathbb{R}^{N_x(N_y-1)\times N_x N_y}$, $A_r \in \mathbb{R}^{(N_y-1)N_y}$, $g_r \in \mathbb{R}^{N_x N_y \times 2N_y}$, $B_n \in \mathbb{R}^{N_y \times N_y}$, $\tilde{g}_r \in \mathbb{R}^{N_x N_y \times 2N_x}$, $C_r \in \mathbb{R}^{N_y \times 2}$,

$$D_r = \frac{1}{\Delta x}\begin{bmatrix} -a_{1,r} & 0 & \cdots & 0 & a_{1,r} & \cdots & a_{r,r} & & \\ \vdots & \ddots & \ddots & & \ddots & \ddots & & \ddots & \\ -a_{r,r} & & \ddots & \ddots & & \ddots & \ddots & & a_{r,r} \\ & \ddots & & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & -a_{r,r} & \cdots & -a_{1,r} & 0 & \cdots & 0 & a_{1,r} \end{bmatrix}, \quad \tilde{D}_r = \frac{1}{\Delta y}\begin{bmatrix} A_r & & \\ & \ddots & \\ & & A_r \end{bmatrix},$$

$$A_r = \begin{bmatrix} -a_{1,r} & a_{1,r} & \cdots & a_{r,r} & & \\ \vdots & \ddots & \ddots & & \ddots & \\ -a_{r,r} & & \ddots & \ddots & & a_{r,r} \\ & \ddots & & \ddots & \ddots & \vdots \\ & & -a_{r,r} & \cdots & -a_{1,r} & a_{1,r} \end{bmatrix}, \quad g_r = \frac{1}{\Delta x}\begin{bmatrix} -B & 0 \\ 0 & \text{flip}(B) \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix}, \quad B_n = b_n I_{N_y}, \quad \tilde{g}_r = \frac{1}{\Delta y}\begin{bmatrix} C_r & & \\ & \ddots & \\ & & C_r \end{bmatrix}, \quad C_r = \begin{bmatrix} -b_1 & 0 \\ \vdots & \vdots \\ -b_r & 0 \\ 0 & b_r \\ \vdots & \vdots \\ 0 & b_1 \end{bmatrix}$$

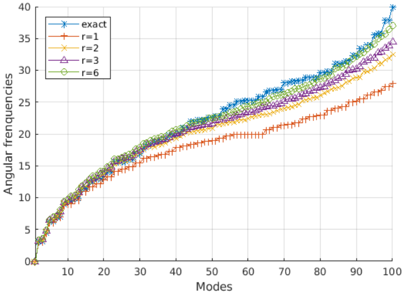
where $b_n = \sum_{i=n}^{r} a_{r,i}$. and $\text{flip}(B) := \left(B_r, ..., B_1\right)^T$.

- ☺ We can reach higher accuracy without increasing the size of the matrices, but only by making it less sparse by increasing r.
- ☹ Any change of the order r requires the recomputation of all the coefficient $a_{n,r}$.

Simulations results and analysis

The 2D Wave Equation as PHS simulated by S-P FVM (2^{nd} order) and Symp Integration (2^{nd} order)

Simulations results and analysis



Continuous Time - Discrete Space

The exact angular frequencies of the continuous operator $\mathcal{J}$ compared to the eigenvalues of discrete operator J_r^d with different values of r.

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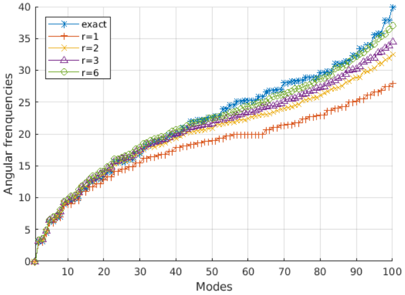

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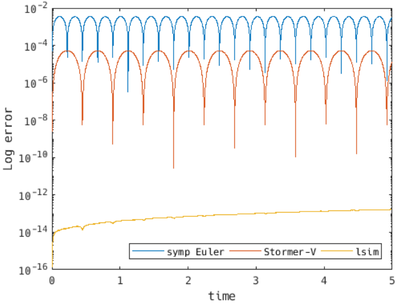

Discrete Time - Discrete Space

Relative error of discrete Hamiltonian, of three different time integration schemes:

- Symplectic:
  - Symplectic Euler (order 1)
  - Stormer-Verlet (order 2)
- Non-symplectic:
  - *lsim*

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Selected References

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PDE simulated by PHS

- Following the causality that we choose for the port-Hamiltonian system, which is

$$e_\partial := \begin{pmatrix} q^1|_{\partial\Omega_x\times\Omega_y} \\ q^2|_{\Omega_x\times\partial\Omega_y} \end{pmatrix} := \begin{pmatrix} \frac{\partial}{\partial x} u|_{\partial\Omega_x\times\Omega_y} \\ \frac{\partial}{\partial y} u|_{\Omega_x\times\partial\Omega_y} \end{pmatrix} = 0 \xrightarrow{\text{Homogenous Neummann}} \nabla u \cdot \vec{\eta}\big|_{\partial\Omega} = 0$$

- To be well-posed, we need to add some initial data.

Full PDE

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(x,y,t) = \Delta u(x,y,t) & \text{in } \Omega \times [0,T], \\ \nabla u(x,y,t)\cdot\vec{\eta}\big|_{\partial\Omega} = 0, & \text{on } \partial\Omega \times [0,T], \\ u(x,y,0) = u_0(x,y) & \text{in } \Omega \times \{0\}, \\ \frac{\partial}{\partial t} u(x,y,0) = 0 & \text{in } \Omega \times \{0\}, \end{cases}$$

- $u_0(x,y)$ is a function satisfying $\nabla u_0 \cdot \vec{\eta}\big|_{\partial\Omega} = 0$.
- Simple example we set $u_0(x,y) = \cos\left(\frac{2n\pi}{l_x} x\right) \cos\left(\frac{2m\pi}{l_y} y\right)$, for arbitrary values of n and m.