Stability of Linear Fractional Differential Equations with Delays

A coupled Parabolic-Hyperbolic PDEs formulation

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Outline

Motivation: fractional delay systems in aeroacoustics

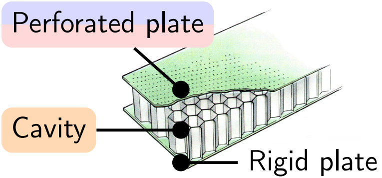
Context: Noise regulations $\Rightarrow$ research into sound absorption.

Modelling of locally-reacting sound absorbing material

Passive LTI system:

$$p(t,x) = \left[z \underset{t}{\star} \boldsymbol{u} \cdot \boldsymbol{n}(\cdot, x)\right](t)$$

with kernel $z \in \mathcal{D}'_+(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$.



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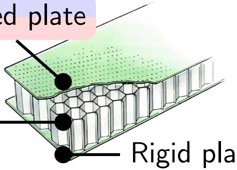

Key components of z: (Monteghetti et al. 2016)

$$\hat{z}(s) = a_0 + a_{1/2}\sqrt{s} + a_\tau e^{-s\tau}$$

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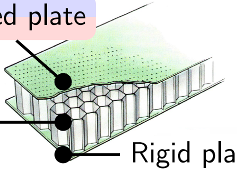

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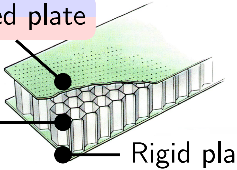

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$\Rightarrow$ Spatial discretisation yields fractional delay equation ($x \in \mathbb{R}^n$):

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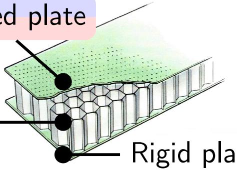

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Objective: use parabolic-hyperbolic realisations to study stability.

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1 Introduction

2 Coupled PDEs formulation: stability results
- Existing stability results
- Scalar "toy" model
- Vector-valued model

3 An eigenvalue approach to stability

4 Conclusion

Existing stability results

Time-delay system of retarded type

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t - \tau_i) \Leftrightarrow \dot{X} = \mathcal{A} X$$

- Roots of characteristic equation $\det \Delta(\lambda) = 0 \Leftrightarrow \lambda \in \sigma_p(\mathcal{A})$.
  (Michiels and Niculescu 2014, Chap. 1) (Curtain and Zwart 1995, § 2.4)
- Lyapunov-Krasovkii equivalence theorem.
  (Fridman 2014, Chap. 3) (Briat 2014, Thm. 5.2.9)

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Fractional differential equation

- BIBO & asymptotic stability with commensurate fractional powers. (Matignon 1996)

Fractional delay differential equation

- BIBO stability with commensurate delays.
  (Bonnet and Partington 2002)
- Asymptotic stability with non-commensurate delays.
  (Deng, Li and Lü 2007)

Toy model: Laplace technique

Objective: delay-independent stability of

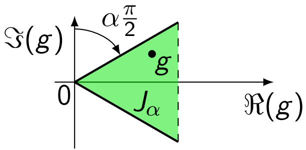
$$\dot{x}(t) = a\,x(t) + b\,x(t-\tau) - g\,d_C^{\alpha}x(t) \quad \text{for } t > \tau,\ \alpha \in (0,1)$$
$$x(t) := x^0(t) \quad \text{for } t \in [0,\tau].$$

Theorem. Toy model stability

Under the following algebraic condition:

$$\Re(a) < -|b| \le 0 \quad \text{and} \quad g \in J_\alpha,$$

toy model is delay-independent stable.



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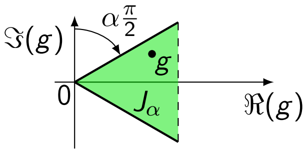

Proof (Sketch). Expression of $\hat{x}(s)$ has 5 terms:

$$\hat{x}(s) = g\,x^0(0)\,\hat{h}_c(s) + g\,x^0(\tau)\,\hat{h}_d(s) + x^0(\tau)\,\hat{h}_e(s)$$
$$+ \widehat{x^0}\,\hat{h}_a(s) + g\,\mathcal{L}[\mathrm{d}_C^{\alpha}x^0\,\mathbb{1}_{[0,\tau]}]\,\hat{h}_b(s).$$

1. $\hat{h}_{c,d,e}(s)$: final-value theorem.
2. $\hat{h}_{a,b}(s)$: Callier-Desoer $\hat{\mathcal{A}}(0)$ class and dominated convergence.

Toy model: coupled PDEs formulation (1)

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Under the algebraic condition

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Proof. Let $E_x := \frac{1}{2}|x|^2$. Decay rate along trajectories is

$$\dot{E}_x = 2\,\Re(a)\,E_x + \Re\Big[\,\overline{x}\,(b\,x_\tau - g\,d_C^{\alpha} x(t))\Big], \tag{1}$$

whose sign is *a priori* indefinite.

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However, energy decay can be proven using suitable realisations.

1. Hyperbolic for $x_\tau(t)$: transport equation.
2. Parabolic for $D_{\mathrm{RL}}^\alpha\,x(t)$: heat equation.
3. Extended energy $\mathcal{E} \Rightarrow$ sufficient condition for decay.

Toy model: coupled PDEs formulation (2)

Hyperbolic realisation $z \in (0, \ell)$ Transport PDE.

(Engel and Nagel 2000, § VI.6) (Curtain and Zwart 1995, § 2.4)

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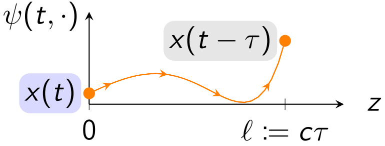

$$\begin{cases} \partial_t \psi(t,z) = -c\,\partial_z \psi(t,z) \\ \psi(t, z=0) := x(t) \\ x(t-\tau) = \psi(t, z=\ell) \end{cases}$$

Initial data $\psi(t=\tau, z) := x^0(\tau - z/c)$.

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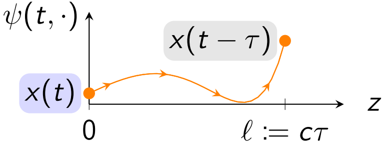

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$$\text{Natural energy:} \quad E_\psi(t) := \frac{1}{2}\int_0^\ell |\psi(t,z)|^2 \, \mathrm{d}z \,.$$

Energy balance reflects lossless transport:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_\psi(t) &= -c \int_0^\ell \Re(\partial_z \psi(t,z)\, \overline{\psi}(t,z))\, \mathrm{d}z \\ &= -\frac{c}{2} \left[\, |\psi(t,z)|^2 \,\right]_0^\ell \\ &= \frac{c}{2}\left(|x(t)|^2 - |x(t-\tau)|^2 \right). \end{aligned}$$

Toy model: coupled PDEs formulation (3)

Parabolic realisation $\xi \in (0, \infty)$ (Parabolic) ODE.

(Staffans 1994) (Montseny 1998) (Matignon 2009) (Hélie and Matignon 2006a)

$$\begin{cases} \partial_t \varphi(\xi, t) = -\xi\, \varphi(\xi, t) + u(t), \ \ \varphi(\xi, 0) = 0\,, \\ D^{\alpha}_{\mathrm{RL}}\, x(t) = \displaystyle\int_0^{\infty} \mu_{1-\alpha}(\xi)\, [u(t) - \xi\, \varphi(\xi, t)]\, \mathrm{d}\xi\,. \end{cases}$$

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Natural energy: $\quad E_\varphi(t) := \dfrac{1}{2} \displaystyle\int_0^\infty \xi\, |\varphi(\xi, t)|^2\, \mu_{1-\alpha}(\xi) \mathrm{d}\xi.$

Energy balance (expresses dissipativity of D^{α}_{RL}):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_\varphi(t) &= \Re(\overline{x}\; D^{\alpha}_{\mathrm{RL}}\, x) - \int_0^\infty |x - \xi\, \varphi(\xi, \cdot)|^2\, \mu_{1-\alpha}(\xi)\, \mathrm{d}\xi\,. \\ &\leq \Re(\overline{x}\; D^{\alpha}_{\mathrm{RL}}\, x). \end{aligned}$$

Toy model: coupled PDEs formulation (4)

Extended energy $\mathcal{E}_k := E_x(t) + k\, E_\psi(t) + g\, E_\varphi(t)$,
with $k > 0$ unknown.

- Parabolic realisation: cross terms $g\, \Re(\overline{x}\, D^\alpha_{\mathrm{RL}} x)$ cancel out

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- Hyperbolic realisation leads to

$$\dot{\mathcal{E}}_k \quad \leq \quad -X^H \Sigma_k X$$

where $X := (x, x_\tau)^\intercal$ and

$$\Sigma_k := -\begin{pmatrix} \Re(a) + k\frac{c}{2} & \frac{b}{2} \\ \frac{\overline{b}}{2} & -k\frac{c}{2} \end{pmatrix} \overset{?}{>} 0.$$

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- Eigenvalues: $\lambda_2(k) > \lambda_1(k) = -\frac{\Re(a)+\sqrt{P(k)}}{2}$, with $P(k) > 0$.
- Least stringent condition:

$$\min_{k>0} \lambda_1(k) = -\frac{\Re(a) + |b|}{2} \quad \text{for} \quad k^* = -\frac{\Re(a)}{c}.$$

$$\lambda_1 > 0 \iff \Re(a) < -|b|$$

Vector-valued model

Vector-valued fractional system with delay:

$$\dot{x}(t) = Ax(t) + B\,x(t-\tau) - G\,d_C^\alpha x(t) \quad \text{for } t > \tau,\ \alpha \in (0,1)$$
$$x(t) := x^0(t) \quad \text{for } t \in [0,\tau],$$

with $x(t) \in \mathbb{R}^n$.

Theorem. Stability.

Let G be a diagonalisable matrix with eigenvalues $(g_1, \cdots, g_n) \geq 0$. Under the algebraic condition

$$\max_{a\in\sigma(A)} \Re(a) < -\sqrt{\max_{b\in\sigma(B^H B)} |b|} \leq 0,$$

the system with $x^0(0) = 0$ is delay-independent stable.

Proof. Similar in spirit to toy model, with extended energy

$$\mathcal{E}(t) := \sum_{i\in[\![1,n]\!]} E_{x_i}(t) + k\,E_{\psi_i}(t) + g_i\,E_{\varphi_i}(t).$$

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2. Coupled PDEs formulation: stability results
3. An eigenvalue approach to stability
   - State of the art
   - Eigenvalue approach to stability
4. Conclusion

Numerical methods for stability: state of the art

Time-delay system of retarded type

- Design an approximate Lyapunov-Krasovkii functional, and formulate a numerically-tractable LMI.

  (Seuret, Gouaisbaut and Ariba 2015) (Baudouin, Seuret and Safi 2016)

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  (Seuret, Gouaisbaut and Ariba 2015) (Baudouin, Seuret and Safi 2016)
- Study of characteristic roots. (Michiels and Niculescu 2014, § 2)
  - Count unstable roots. (Li, Niculescu and Cela 2015)
  - Locate unstable roots : eigenvalue approach.
    - Spectrum of operator semigroup $e^{t\mathcal{A}}$.
      DDE-BIFTOOL (Engelborghs, Luzyanina and Roose 2002)
    - Spectrum of generator $\mathcal{A}$ using hyperbolic realisation.
      TRACE-DDE (Breda, Maset and Vermiglio 2005)

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Fractional delay differential equation

- YALTA. (Fioravanti et al. 2012) (Avanessoff, Fioravanti and Bonnet 2013)
- Count unstable roots. (Zhang et al. 2016)

$\Rightarrow$ Eigenvalue approach using parabolic-hyperbolic realisation?

Eigenvalue approach to stability: overview

Vector-valued fractional delay system:

$$\dot{x}(t) = A\,x(t) + B\,x(t-\tau) - G\,d_C^{\alpha}x(t-\tau_\alpha) \quad (\tau_\alpha \geq 0).$$

Hyperbolic realisation (PDE)

$$z \in (0,1)$$

$$\partial_t \psi_h = -\tau^{-1}\partial_z \psi_h$$
$$\psi_h(0) = x$$
$$x(t-\cdot) = \psi_h(z=1)$$

$\Rightarrow$ High-order discretisation

Parabolic realisation (ODE)

$$\xi \in (0,\infty)$$

$$\partial_t \varphi_h = -\xi\varphi_h + x$$
$$d_C^{\alpha}x = \sum_{k\in[\![1,N_\xi]\!]} \mu_k\,\varphi_h(\xi_k)$$

$\Rightarrow$ Quadrature or optimisation

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Hyperbolic realisation (PDE)	Parabolic realisation (ODE)
$z \in (0,1)$	$\xi \in (0,\infty)$
$\partial_t \psi_h = -\tau^{-1}\partial_z \psi_h$ $\psi_h(0) = x$ $x(t-\cdot) = \psi_h(z=1)$	$\partial_t \varphi_h = -\xi\varphi_h + x$ $d_C^{\alpha}x = \sum_{k\in[\![1,N_\xi]\!]} \mu_k\,\varphi_h(\xi_k)$
$\Rightarrow$ High-order discretisation	$\Rightarrow$ Quadrature or optimisation

$\Rightarrow$ Cauchy problem on $\mathbb{C}^n$:

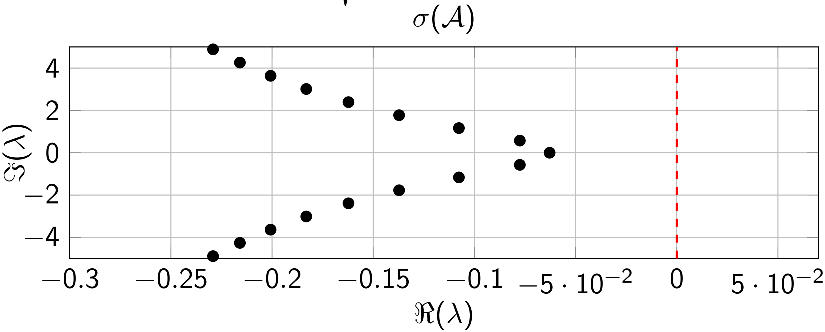
$$\dot{X}_h(t) = \mathcal{A}_h\,X_h(t), \quad \text{with } X_h := (x, \psi_h, \varphi_h).$$

Challenge: ensure $\sigma(\mathcal{A}_h)$ is "meaningful".

Numerical experiment: spectral structure

Case 1: $x(t) \in \mathbb{R}^2$, $\dot{x}(t) = A \cdot x(t) + B \cdot x(t-\tau) - g\, I_2 \cdot d_C^{1/2} x(t)$,
with

$$\max_{a \in \sigma(A)} \Re(a) < -\sqrt{\max_{b \in \sigma(B^H B)} |b|} \leq 0 \quad \text{verified.}$$



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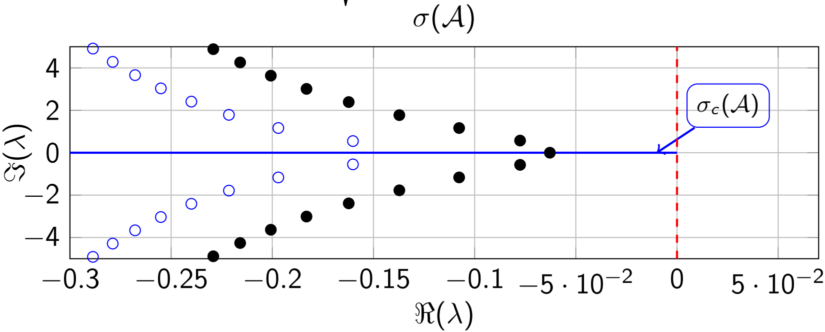

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Fractional derivative $g \neq 0 \Rightarrow \sigma_c(\mathcal{A}) \neq \emptyset$ (essential)

(∘) $g = +2 > 0 \Rightarrow$ stable

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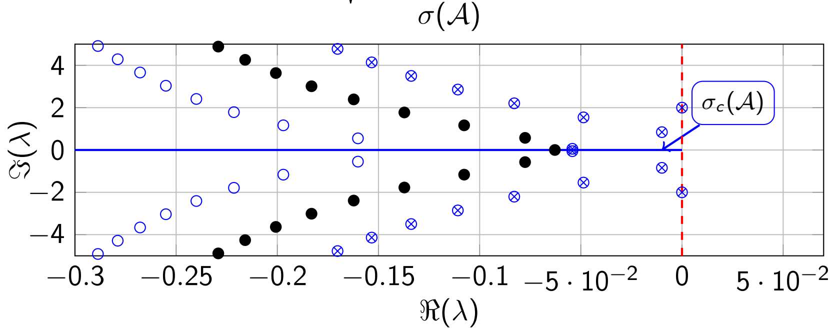

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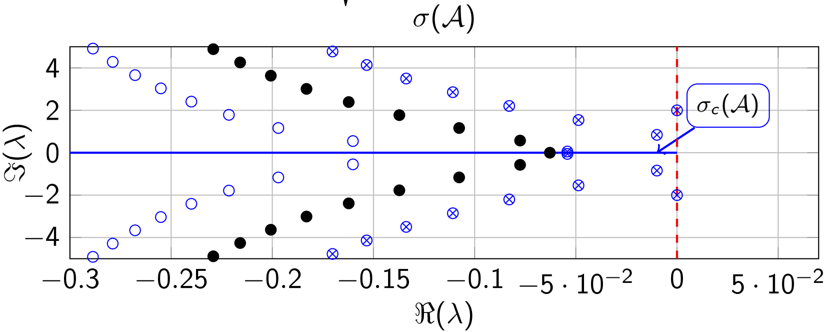
Fractional derivative $g \neq 0 \Rightarrow \sigma_c(\mathcal{A}) \neq \emptyset$ (essential)

(○) $g = +2 > 0 \Rightarrow$ stable (⊗) $g = -2 < 0 \Rightarrow$ unstable

Numerical experiment: spectral structure

Case 1: $x(t) \in \mathbb{R}^2$, $\dot{x}(t) = A \cdot x(t) + B \cdot x(t-\tau) - g\, I_2 \cdot d_C^{1/2} x(t)$, with

$$\max_{a\in\sigma(A)} \Re(a) < -\sqrt{\max_{b\in\sigma(B^H B)} |b|} \leq 0 \quad \text{verified.}$$



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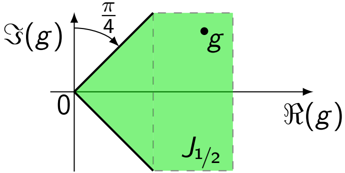
Fractional derivative $g \neq 0 \Rightarrow \sigma_c(\mathcal{A}) \neq \emptyset$ (essential)

(○) $g = +2 > 0 \Rightarrow$ stable (⊗) $g = -2 < 0 \Rightarrow$ unstable

What about $g \in \mathbb{C}$?

Numerical experiment: delay-dependent stability

Case 2: Scalar model $\dot{x}(t) = -x(t) + \frac{1}{2}\,x(t-\tau) - g\,d_C^{1/2}x(t)$.
For delay-independent stability, $g \in \boxed{J_{1/2}}$.



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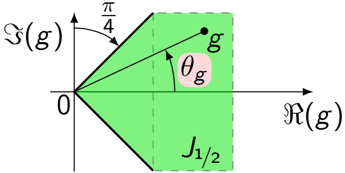
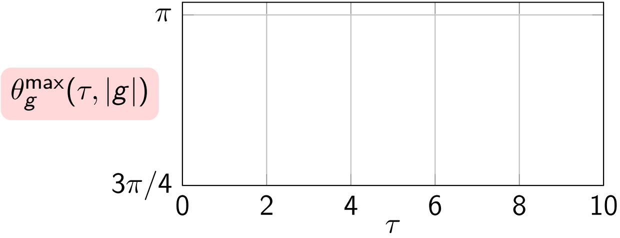


What about the delay-dependent stability region?



Numerical experiment: delay-dependent stability

Case 2: Scalar model $\dot{x}(t) = -x(t) + \frac{1}{2}x(t-\tau) - g\, d_C^{1/2} x(t)$.
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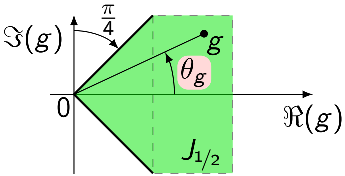

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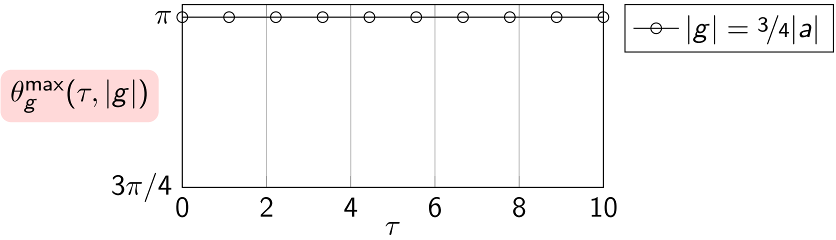

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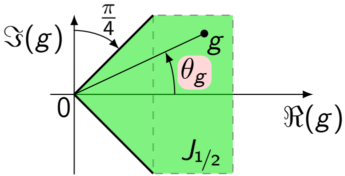

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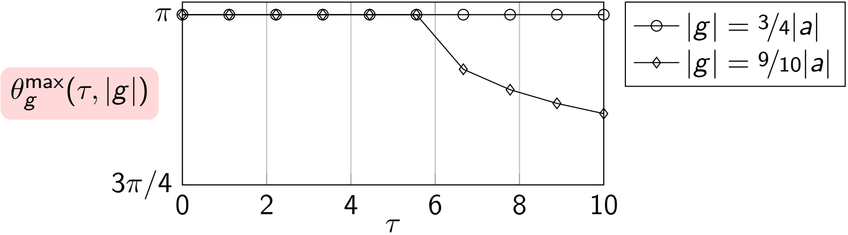

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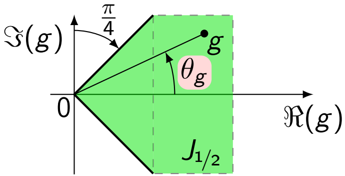

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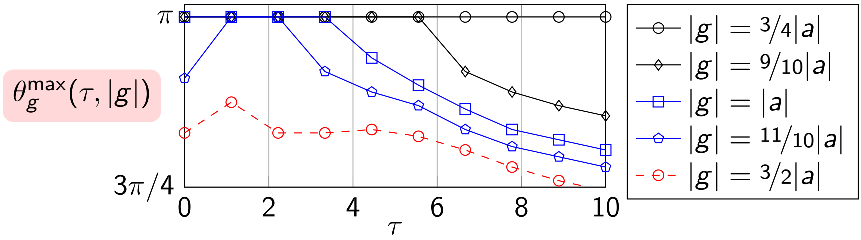
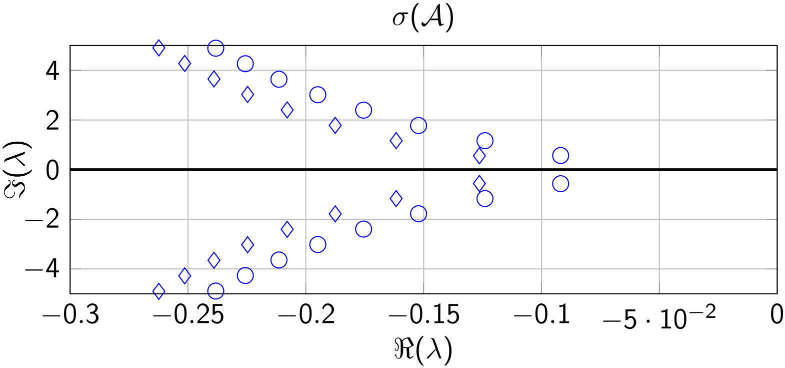


Numerical experiment: composition (exploratory)

Case 3: Scalar model $\dot{x}(t) = -x(t) + \frac{1}{2}x(t-\tau) - g\, d_C^{1/2} x(t - \tau_\alpha)$.



Effect of delaying the fractional derivative

$g = |a|/4$.

1. $\tau_\alpha = 0$ (○).

$g = |a|$.

1. $\tau_\alpha = 0$ (◊).

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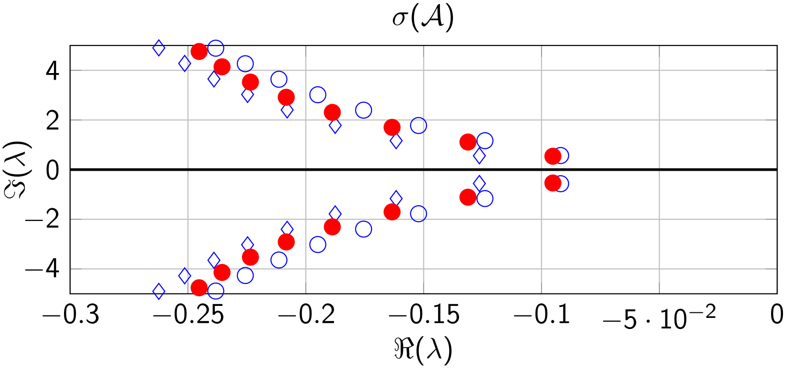

Effect of delaying the fractional derivative

$g = |a|/4$.

1. $\tau_\alpha = 0$ (○).
2. $\tau_\alpha = \tau$ (●).

$g = |a|$.

1. $\tau_\alpha = 0$ (◇).

Numerical experiment: composition (exploratory)

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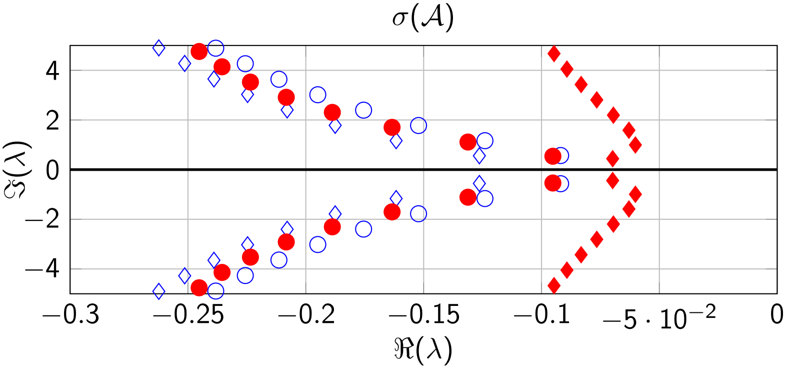

Effect of delaying the fractional derivative

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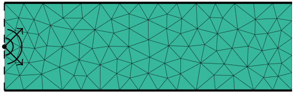
1. $\tau_\alpha = 0$ (◇).
2. $\tau_\alpha = \tau$ (♦).

Application in acoustics

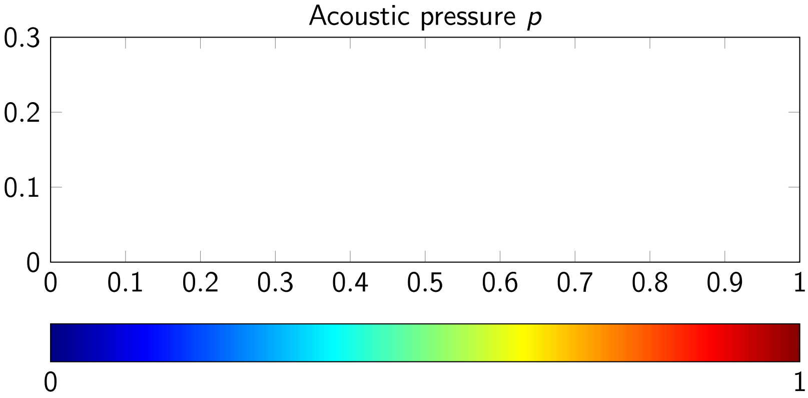
Computational case Infinite 2D duct.

DG: $N = 4$. Mesh: $N_K = 188$.

Time-integration: CFL $= 0.5$. (LSERK (8,4) (Toulorge and Desmet 2012))



$\hat{z}(s, x) = \infty$ (Rigid Wall)



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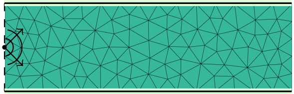
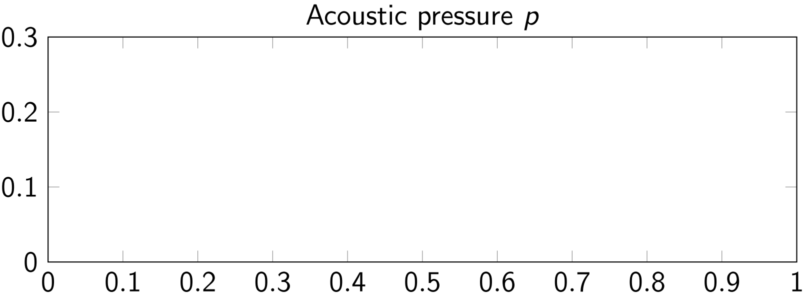


$$\hat{z}(s) = a + a\sqrt{s} + \frac{a}{2}e^{-s\tau} \quad \text{(Soft Wall)}$$



Outline

Conclusion

Takeaways

- Parabolic-Hyperbolic PDE realisations $\Rightarrow$ time-local coupled system (x, φ, ψ)
- Natural extended energy $\mathcal{E} = E_x + E_\varphi + k\, E_\psi$

  $\Rightarrow$ sufficient asymptotic stability condition
  $\Rightarrow$ eigenvalue approach to stability
- Application to aeroacoustics

Perspectives

- Multiple delay case
- Semigroup formulation
- Composition: $D^\alpha_{\mathrm{RL}} x(t-\tau)$?
- Theoretical study of eigenvalue approach

Conclusion

1. Introduction
2. Coupled PDEs formulation: stability results
3. An eigenvalue approach to stability
4. Conclusion

▸ Appendix

Thanks for your attention. Any questions?

(Contact: `florian.monteghetti@onera.fr`)

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Additional slides

◂ Main presentation

Outline

5. Coupled formulation: stability results
   - Toy model: Laplace analysis
   - Vector-valued case
   - Parabolic realisation and heat equation
6. An eigenvalue approach to stability

Toy model: Laplace analysis

$$\hat{x}(s) = \underbrace{(s-a)\hat{h}(s)\,\widehat{x^0}(s)}_{\text{(a)}} + g\,\underbrace{\hat{h}(s)\,\mathcal{L}_0[d_C^\alpha x^0\,\mathbb{1}_{[0,\tau]}](s)}_{\text{(b)}}$$
$$+\,gx^0(0)\,\underbrace{\hat{h}(s)\,s^\alpha}_{\text{(c)}} + gx^0(\tau)\,\underbrace{\hat{h}(s)s^\alpha e^{-s\tau}}_{\text{(d)}}$$
$$+\,x^0(\tau)\,\underbrace{\hat{h}(s)e^{-s\tau}}_{\text{(e)}},$$

where $\hat{h}(s) := (s - a - b\,\exp(-\tau\, s) + g\, s^\alpha)^{-1}$. $\hat{h}$ is defined over $\mathbb{C}^+_\beta$ for $\beta \geq 0$ (cut).

Toy model: Laplace analysis (2)

$\hat{h}(s) := (s - a - b\exp(-\tau s) + g\, s^{\alpha})^{-1}$ is defined over $\mathbb{C}^+_\beta$ for $\beta \geq 0$ (cut).
Analyticity? Let $s = x + i\, y = r\exp(i\theta)$ with $\{x \geq 0$ and $y \in \mathbb{R}\}$ or $\{r \geq 0$ and $|\theta| \leq \frac{\pi}{2}\}$.

$$\begin{aligned}\Re(s - a - be^{-\tau s} + gs^{\alpha}) &= x - \Re(a) - |b|e^{-\tau x}\cos(\tau y - \theta_b) + \Re(gs^{\alpha}) \\ &\geq x - \Re(a) - |b| + |g||s|^{\alpha} \\ &\geq -\Re(a) - |b|,\end{aligned}$$

using $\theta_g \in J_\alpha$. If $\Re(a) < -|b|$, then $\hat{h}$ analytic in $\overline{\mathbb{C}^+_0}$.

Toy model: Laplace analysis (3)

If $\theta_g \in J_\alpha$ and $\Re(a) < -|b|$,
$\hat{h}(s) := (s - a - b\,\exp(-\tau\, s) + g\, s^\alpha)^{-1} \in \widehat{\mathcal{A}}(0)$, the Callier-Desoer class. (Curtain and Zwart 1995, Thm. A.7.49)

- $(s-a)^{-1} \in \widehat{\mathcal{A}}(0)$, since $\Re(a) < 0$. It remains to show that
$$[1 - (b\,\exp(-\tau\, s) - g\, s^\alpha)/(s-a)]^{-1} \in \widehat{\mathcal{A}}(0).$$
We first notice that
$$\hat{f}(s) = 1 - (b\,\exp(-\tau\, s) - g\, s^\alpha)/(s-a) \in \widehat{\mathcal{A}}(0).$$
Then, it is enough to prove that $\inf_{\Re(s)\geq 0} |\hat{f}(s)| > 0$ to ensure that $1/\hat{f} \in \widehat{\mathcal{A}}(0)$.
- As a consequence, $h \in \mathcal{A}(0)$, i.e. can be decomposed into
$$h(t) = h_a(t) + \sum_{n\in\mathbb{N}} h_n\,\delta(t - t_n)\,, \tag{2}$$
with $h_a \in L^1(\mathbb{R}^+)$ and $(h_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$; and $0 = t_0 < t_1 < t_2 < \cdots$.

Vector-valued case

The energy balance reads

$$\dot{\mathcal{E}} = \dot{x} \cdot x + \frac{kc}{2}[\|x\|^2 - \|x(\cdot - \tau)\|^2] + \sum_{i \in [\![1,n]\!]} g_i \dot{E}_{\widetilde{\varphi}_i}.$$

G diagonal $\Rightarrow$ no coupling btw diffusive variables $\widetilde{\varphi}_i$: delay/fractional coupling is straightforward.
Since $g_i \geq 0$, it is sufficient for to prove that

$$-\Sigma_k := \begin{bmatrix} \frac{A + A^H}{2} + \frac{kc}{2} I & \frac{1}{2} B \\ \frac{1}{2} B^H & -\frac{kc}{2} I \end{bmatrix} < 0,$$

Let us denote $A^S = {}^{(A+A^H)}/_2$ the symmetric part of A. We have for all x and y in $\mathbb{C}^n$

$$-\Sigma_k \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A^S x \cdot x + \frac{kc}{2}\|x\|^2 + \frac{1}{2} B y \cdot x + \frac{1}{2} B^H x \cdot y - \frac{kc}{2}\|y\|^2.$$

Now, for any $\varepsilon > 0$, from

Vector-valued case (2)

Now, for any $\varepsilon > 0$, from

$$\frac{1}{2}By \cdot x + \frac{1}{2}B^H x \cdot y = \Re\left(By \cdot x\right) \leq \frac{\varepsilon}{2}\|By\|^2 + \frac{1}{2\varepsilon}\|x\|^2,$$

and

$$\|By\|^2 = B^H By \cdot y \leq \max_{b \in \sigma(B^H B)} |b| \|y\|^2,$$

we can choose $k^* = k_\varepsilon = {}^{\varepsilon \max_{b \in \sigma(B^H B)} |b|}/_{c} > 0$ to get

$$-\Sigma_k \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \leq \left(\max_{a \in \sigma(A)} \Re(a) + \frac{k_\varepsilon c}{2} + \frac{1}{2\varepsilon} \right) \|x\|^2.$$

Taking the least stringent value of $\varepsilon > 0$, we derive

$$\max_{a \in \sigma(A)} \Re(a) + \sqrt{\max_{b \in \sigma(B^H B)} |b|} < 0$$

as a delay-independent stability sufficient condition.

Long-memory impedance: link with heat equation

Monodimensional heat equation

(Matignon and Zwart in revision, Ex. 3.2)

$$\begin{cases} \partial_t \theta(t,x) & = \partial_x^2 \theta(t,x) + u(t)\, \delta_0(x) \quad (x \in \mathbb{R}, t > 0) \\ y(t) & = 4\, \theta(t, x=0). \end{cases}$$

Take the Fourier and Laplace transform

$$\hat{\theta}(s,k) := \int_0^\infty \int_{\mathbb{R}} \theta(t,x)\, e^{-ikx} e^{-st}\, \mathrm{d}x\, \mathrm{d}t \quad (\Re[s] > 0, k \in \mathbb{R})$$

to get

$$\hat{\theta}(s,k) = \frac{1}{s+k^2} \hat{u}(s),$$

hence

$$\hat{y}(s) = 4 \times \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\theta}(s,k)\, \mathrm{d}k = \underbrace{\left(\int_{\mathbb{R}} \frac{1}{s+k^2} \frac{2\mathrm{d}k}{\pi} \right)}_{=1/\sqrt{s}} \hat{u}(s).$$

Outline

Discretisation of parabolic representation

Discretisation problem (Laplace domain):

$$\forall \Re(s) > 0, \quad \hat{h}(s) = \int_0^\infty \frac{1}{s+\xi}\,\mathrm{d}\mu(\xi) \simeq \sum_{k \in [\![1, N_\xi]\!]} \mu_k \frac{1}{s+\xi_k} = \hat{h}_{\mathrm{num}}(s).$$

Challenges:

- Parsimonious approximation
- Spectral accuracy (no spurious instabilities)
- Monotone convergence for $N_\xi \to \infty$

Discretisation of parabolic representation (2)

Discretisation problem (Laplace domain):

$$\forall \Re(s) > 0, \quad \hat{h}(s) = \int_0^{\infty} \frac{1}{s+\xi}\,\mathrm{d}\mu(\xi) \simeq \sum_{k \in [\![1, N_\xi]\!]} \mu_k \frac{1}{s+\xi_k} = \hat{h}_{\mathrm{num}}(s).$$

Methods:

- Interpolation (Heleschewitz 2000)
- Optimisation (Hélie and Matignon 2006b, SP).

$$J(\boldsymbol{\xi}, \boldsymbol{\mu}) = \int_{\omega_{\min}}^{\omega_{\max}} \left|\hat{h}(i\omega) - \hat{h}_{\mathrm{num}}(i\omega)\right|^2 w(\omega)\,\mathrm{d}\omega.$$

Parameters: $\xi_{\min}$, $\xi_{\max}$, N_ξ and $\omega_{\min}$, $\omega_{\max}$ and w.

$\Rightarrow$ Quadrature (analytical expressions for μ_k and ξ_k).

$$\int_0^{\infty} \mu_{\overline{\alpha}}(\xi)\,\varphi(\xi, t)\,\mathrm{d}\xi \quad = \quad \int_0^1 \mu_{\overline{\alpha}}(\Phi(v))\,\varphi(\Phi(v), t)\,\Phi'(v)\,\mathrm{d}v,$$

with $\Phi(v) := v^2(1-v)^{-2}$ for regularity at $\xi = 0$, see (Shampine 2008, S 4.2). Then Gauss-Legendre quadrature rule.