Recovering the initial state of a Well-Posed Linear System with perturbed skew-adjoint generator

Ghislain Haine

ISAE – Supported by IDEX-"Nouveaux Entrants"

European Control Conference July, 15–17

Session "State Observation and Parameter Estimation of Systems Involving PDEs"

Let

- X be a Hilbert space,
- $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$ be a skew-adjoint operator,

Let

- X be a Hilbert space,
- $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$ be a skew-adjoint operator,

Considered systems: Given $\alpha \in \mathbb{R}$, let $A = \mathcal{A} + \alpha I$,

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \in [0, \infty), \\ z(0) = z_0 \in \mathcal{D}(\mathcal{A}). \end{cases}$$

Let

- X be a Hilbert space,
- $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \to X$ be a skew-adjoint operator,

Considered systems: Given $\alpha \in \mathbb{R}$, let $A = \mathcal{A} + \alpha I$,

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \in [0, \infty), \\ z(0) = z_0 \in \mathcal{D}(\mathcal{A}). \end{cases}$$

For instance:

$\mathcal{A} = -\mathbf{i}\Delta$ (+ Dirichlet boundary conditions) on $\Omega \subset \mathbb{R}^n$ and $X = H_0^1(\Omega)$.

$$\Downarrow$$

the classical Schrödinger's equation with constant potential.

Let

- Y be another Hilbert space
- $C \in \mathcal{L}(X, Y)$
- $\tau > 0$

Let

- Y be another Hilbert space
- $C \in \mathcal{L}(X, Y)$
- $\tau > 0$

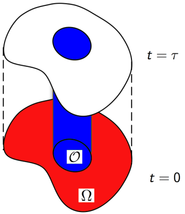
We observe z *via* $y(t) = Cz(t)$ for all $t \in [0, \tau]$.

Let

- Y be another Hilbert space
- $C \in \mathcal{L}(X, Y)$
- $\tau > 0$

We observe z *via* $y(t) = Cz(t)$ for all $t \in [0, \tau]$.

The Schrödinger's equation case,
with locally distributed observation:



$$C = \chi_{\mathcal{O}} \cdot$$

and

$$y(t) \quad = \chi_{\mathcal{O}} z(t), \quad \forall t \in [0, \tau].$$

Let

- Y be another Hilbert space
- $C \in \mathcal{L}(X, Y)$
- $\tau > 0$

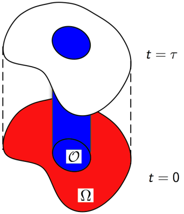
We observe z *via* $y(t) = Cz(t)$ for all $t \in [0, \tau]$.

The Schrödinger's equation case, with locally distributed observation:

$$C = \chi_{\mathcal{O}} \cdot$$

and

$$y(t) = \chi_{\mathcal{O}} z(t), \quad \forall t \in [0, \tau].$$

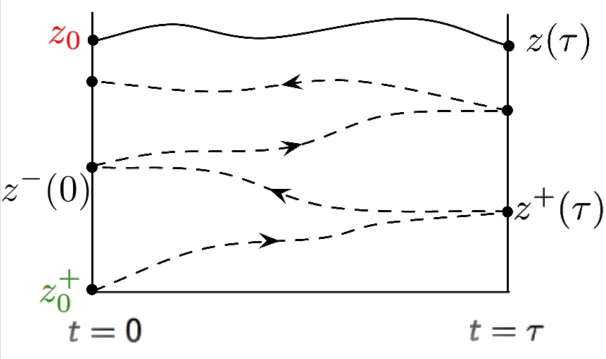


Our problem

Reconstruct the unknown z_0 from the measurement $y(t)$.

K. Ramdani, M. Tucsnak, and G. Weiss
Recovering the initial state of an infinite-dimensional system using observers (Automatica, 2010)

Intuitive representation



2 iterations, observation on $[0, \tau]$.

We construct the **forward observer**

$$\begin{cases} \dot{z}^+(t) = Az^+(t) - C^*Cz^+(t) + C^*y(t), & \forall\, t \in [0,\tau], \\ z^+(0) = z_0^+ \in \mathcal{D}(A). \end{cases}$$

We construct the **forward observer**

$$\begin{cases} \dot{z}^+(t) = Az^+(t) - C^*Cz^+(t) + C^*y(t), & \forall\, t \in [0,\tau], \\ z^+(0) = z_0^+ \in \mathcal{D}(A). \end{cases}$$

We subtract the observed system

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \in [0,\tau], \\ z(0) = z_0, \end{cases}$$

We construct the **forward observer**

$$\begin{cases} \dot{z}^+(t) = Az^+(t) - C^*Cz^+(t) + C^*y(t), & \forall\, t \in [0,\tau], \\ z^+(0) = z_0^+ \in \mathcal{D}(A). \end{cases}$$

We subtract the observed system

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \in [0,\tau], \\ z(0) = z_0, \end{cases}$$

to obtain *(remember that $y(t) = Cz(t)$)*, denoting

$$e = z^+ - z,$$

the estimation error,

$$\begin{cases} \dot{e}(t) = (A - C^*C)\, e(t), & \forall\, t \in [0,\tau], \\ e(0) = z_0^+ - z_0, \end{cases}$$

We construct the **forward observer**

$$\begin{cases} \dot{z}^+(t) = Az^+(t) - C^*Cz^+(t) + C^*y(t), & \forall\, t \in [0,\tau], \\ z^+(0) = z_0^+ \in \mathcal{D}(A). \end{cases}$$

We subtract the observed system

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \in [0,\tau], \\ z(0) = z_0, \end{cases}$$

to obtain *(remember that $y(t) = Cz(t)$)*, denoting

$$e = z^+ - z,$$

the estimation error,

$$\begin{cases} \dot{e}(t) = (A - C^*C)\, e(t), & \forall\, t \in [0,\tau], \\ e(0) = z_0^+ - z_0, \end{cases}$$

which is known to be exponentially stable if and only if (A, C) is exactly observable, *i.e.*

$$\exists \tau > 0, \exists k_\tau > 0, \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(A).$$

Exponential stability $\Rightarrow \exists M > 0, \beta > 0$ such that

$$\|z^+(\tau) - z(\tau)\| \leq M e^{-\beta\tau} \|z_0^+ - z_0\|.$$

Exponential stability $\Rightarrow \exists M > 0, \beta > 0$ such that

$$\|z^+(\tau) - z(\tau)\| \leq Me^{-\beta\tau}\|z_0^+ - z_0\|.$$

We construct a similar system: the **backward observer**,

$$\begin{cases} \dot{z}^-(t) = Az^-(t) + C^*Cz^-(t) - C^*y(t), & \forall\, t \in [0, \tau], \\ z^-(\tau) = z^+(\tau). \end{cases}$$

Exponential stability $\Rightarrow \exists M > 0, \beta > 0$ such that

$$\|z^+(\tau) - z(\tau)\| \leq M e^{-\beta\tau} \|z_0^+ - z_0\|.$$

We construct a similar system: the **backward observer**,

$$\begin{cases} \dot{z}^-(t) = Az^-(t) + C^*Cz^-(t) - C^*y(t), & \forall\, t \in [0, \tau], \\ z^-(\tau) = z^+(\tau). \end{cases}$$

After a time reversal $Z^-(t) = Я_\tau z^-(t) := z^-(\tau - t)$, we get

$$\begin{cases} \dot{Z}^-(t) = -AZ^-(t) - C^*CZ^-(t) + C^*y(\tau - t), & \forall\, t \in [0, \tau], \\ Z^-(0) = z^+(\tau). \end{cases}$$

Exponential stability $\Rightarrow \exists M > 0, \beta > 0$ such that

$$\|z^+(\tau) - z(\tau)\| \leq M e^{-\beta\tau} \|z_0^+ - z_0\|.$$

We construct a similar system: the **backward observer**,

$$\begin{cases} \dot{z}^-(t) = Az^-(t) + C^*Cz^-(t) - C^*y(t), & \forall\, t \in [0,\tau], \\ z^-(\tau) = z^+(\tau). \end{cases}$$

After a time reversal $Z^-(t) = Я_\tau z^-(t) := z^-(\tau - t)$, we get

$$\begin{cases} \dot{Z}^-(t) = -AZ^-(t) - C^*CZ^-(t) + C^*y(\tau - t), & \forall\, t \in [0,\tau], \\ Z^-(0) = z^+(\tau). \end{cases}$$

And from similar computations for $A^- := -A - C^*C$ as those for $A^+ := A - C^*C$:

$$\|z^-(0) - z_0\| \leq M e^{-\beta\tau} \|z^+(\tau) - z(\tau)\| \leq M^2 e^{-2\beta\tau} \|z_0^+ - z_0\|.$$

If the system is exactly observable in time $\tau > 0$, that is if:

$$\exists k_\tau > 0,\ \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\ z_0 \in \mathcal{D}(A),$$

Ito, Ramdani and Tucsnak (Discrete Contin. Dyn. Syst. Ser. S, 2011) proved that

$$\alpha := M^2 e^{-2\beta\tau} < 1.$$

If the system is exactly observable in time $\tau > 0$, that is if:

$$\exists k_\tau > 0, \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(A),$$

Ito, Ramdani and Tucsnak (Discrete Contin. Dyn. Syst. Ser. S, 2011) proved that

$$\alpha := M^2 e^{-2\beta\tau} < 1.$$

Iterating n-times the forward–backward observers with $z_n^+(0) = z_{n-1}^-(0)$ leads to

$$\|z_n^-(0) - z_0\| \leq \alpha^n \|z_0^+ - z_0\|.$$

This is the iterative algorithm of Ramdani, Tucsnak and Weiss to reconstruct z_0 from $y(t)$.

In this work, the exact observability assumption in time τ

$$\exists k_\tau > 0,\ \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\ z_0 \in \mathcal{D}(\mathcal{A}),$$

is not supposed to be satisfied !

In this work, the exact observability assumption in time τ

$$\exists k_\tau > 0,\ \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\ z_0 \in \mathcal{D}(\mathcal{A}),$$

is not supposed to be satisfied !

However, the observers don't need this assumption to make sense.

In this work, the exact observability assumption in time τ

$$\exists k_\tau > 0, \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(\mathcal{A}),$$

is not supposed to be satisfied !
However, the observers don't need this assumption to make sense.

Questions

- Given arbitrary C and $\tau > 0$, does the algorithm converge ?
- If it does, what is the limit of $z_n^-(0)$ and how is it related to z_0 ?

Decomposition of X:

- Let us denote Ψ_τ the following continuous linear operator

$$\begin{array}{cccc} \Psi_\tau & : & X & \longrightarrow & L^2\left([0,\tau],Y\right), \\ & & z_0 & \mapsto & y(t). \end{array}$$

Decomposition of X:

- Let us denote Ψ_τ the following continuous linear operator

$$\begin{array}{rcccl} \Psi_\tau & : & X & \longrightarrow & L^2\left([0,\tau], Y\right), \\ & & z_0 & \mapsto & y(t). \end{array}$$

Intuitively, if z_0 is in $\mathrm{Ker}\ \Psi_\tau$, then $y(t) \equiv 0$, and we have no information on z_0 !

Decomposition of X:

- Let us denote Ψ_τ the following continuous linear operator

$$\begin{array}{rcccl} \Psi_\tau & : & X & \longrightarrow & L^2\left([0,\tau],Y\right), \\ & & z_0 & \mapsto & y(t). \end{array}$$

  Intuitively, if z_0 is in $\mathrm{Ker}\ \Psi_\tau$, then $y(t) \equiv 0$, and we have no information on z_0 !

- We decompose $X = \mathrm{Ker}\ \Psi_\tau \oplus \left(\mathrm{Ker}\ \Psi_\tau\right)^\perp$ and define

$$\mathrm{V}_{\mathrm{Unobs}} = \mathrm{Ker}\ \Psi_\tau, \quad \mathrm{V}_{\mathrm{Obs}} = \left(\mathrm{Ker}\ \Psi_\tau\right)^\perp = \overline{\mathrm{Ran}\ \Psi_\tau^*}.$$

Decomposition of X:

- Let us denote Ψ_τ the following continuous linear operator

$$\begin{array}{cccc} \Psi_\tau & : & X & \longrightarrow & L^2([0,\tau],Y), \\ & & z_0 & \mapsto & y(t). \end{array}$$

  Intuitively, if z_0 is in $\mathrm{Ker}\ \Psi_\tau$, then $y(t) \equiv 0$, and we have no information on z_0 !

- We decompose $X = \mathrm{Ker}\ \Psi_\tau \oplus (\mathrm{Ker}\ \Psi_\tau)^\perp$ and define

$$\mathrm{V}_{\mathrm{Unobs}} = \mathrm{Ker}\ \Psi_\tau, \quad \mathrm{V}_{\mathrm{Obs}} = (\mathrm{Ker}\ \Psi_\tau)^\perp = \overline{\mathrm{Ran}\ \Psi_\tau^*}.$$

  Note that the exact observability assumption is equivalent to Ψ_τ is bounded from below and then $\Rightarrow X = \mathrm{Ran}\ \Psi_\tau^*$.

Stability of the decomposition under the algorithm:
Let us denote $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) the semigroup generated by $\mathcal{A}^+ := \mathcal{A} - C^*C$ (resp. $\mathcal{A}^- := -\mathcal{A} - C^*C$) on X.

- Forward–backward observers cycle $\Rightarrow$ operator $\mathbb{T}^-_\tau \mathbb{T}^+_\tau$, *i.e.*

$$z^-(0) - z_0 = \mathbb{T}^-_\tau \mathbb{T}^+_\tau \left(z_0^+ - z_0 \right),$$

  obtained thanks to the fact that $A^\pm = \mathcal{A}^\pm \pm \alpha I$ generates the semigroup $e^{\pm \alpha t} \mathbb{T}^\pm_t$.

Stability of the decomposition under the algorithm:
Let us denote $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) the semigroup generated by $\mathcal{A}^+ := \mathcal{A} - C^*C$ (resp. $\mathcal{A}^- := -\mathcal{A} - C^*C$) on X.

- Forward–backward observers cycle ⇒ operator $\mathbb{T}^-_\tau \mathbb{T}^+_\tau$, *i.e.*

$$z^-(0) - z_0 = \mathbb{T}^-_\tau \mathbb{T}^+_\tau \left(z_0^+ - z_0 \right),$$

  obtained thanks to the fact that $A^\pm = \mathcal{A}^\pm \pm \alpha I$ generates the semigroup $e^{\pm \alpha t} \mathbb{T}^\pm_t$.

- Denote $\mathbb{S}$ the group generated by $\mathcal{A}$, then (since $\mathcal{A} = \mathcal{A}^+ + C^*C$)

$$\mathbb{S}_\tau z_0 = \mathbb{T}^+_\tau z_0 + \int_0^\tau \mathbb{T}^+_{\tau - t} C^* \underbrace{C \mathbb{S}_t z_0}_{\Psi_\tau z_0} dt, \qquad \forall\, z_0 \in X.$$

Stability of the decomposition under the algorithm:
Let us denote $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) the semigroup generated by $\mathcal{A}^+ := \mathcal{A} - C^*C$ (resp. $\mathcal{A}^- := -\mathcal{A} - C^*C$) on X.

- Forward–backward observers cycle $\Rightarrow$ operator $\mathbb{T}_\tau^- \mathbb{T}_\tau^+$, *i.e.*

$$z^-(0) - z_0 = \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \left(z_0^+ - z_0 \right),$$

  obtained thanks to the fact that $A^\pm = \mathcal{A}^\pm \pm \alpha I$ generates the semigroup $e^{\pm \alpha t} \mathbb{T}_t^\pm$.

- Denote $\mathbb{S}$ the group generated by $\mathcal{A}$, then (since $\mathcal{A} = \mathcal{A}^+ + C^*C$)

$$\mathbb{S}_\tau z_0 = \mathbb{T}_\tau^+ z_0 + \int_0^\tau \mathbb{T}_{\tau - t}^+ C^* \underbrace{C \mathbb{S}_t z_0}_{\Psi_\tau z_0} dt, \qquad \forall\ z_0 \in X.$$

- Using this (type of) Duhamel formula(s), we obtain

$$\mathbb{T}_\tau^- \mathbb{T}_\tau^+ \mathrm{V}_{\mathrm{Unobs}} \subset \mathrm{V}_{\mathrm{Unobs}}, \quad \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \mathrm{V}_{\mathrm{Obs}} \subset \mathrm{V}_{\mathrm{Obs}}.$$

Stability of the decomposition under the algorithm:
Let us denote $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) the semigroup generated by $\mathcal{A}^+ := \mathcal{A} - C^*C$ (resp. $\mathcal{A}^- := -\mathcal{A} - C^*C$) on X.

- Forward–backward observers cycle $\Rightarrow$ operator $\mathbb{T}^-_\tau \mathbb{T}^+_\tau$, *i.e.*

$$z^-(0) - z_0 = \mathbb{T}^-_\tau \mathbb{T}^+_\tau \left(z_0^+ - z_0 \right),$$

  obtained thanks to the fact that $A^\pm = \mathcal{A}^\pm \pm \alpha I$ generates the semigroup $e^{\pm\alpha t}\mathbb{T}^\pm_t$.

- Denote $\mathbb{S}$ the group generated by $\mathcal{A}$, then (since $\mathcal{A} = \mathcal{A}^+ + C^*C$)

$$\mathbb{S}_\tau z_0 = \mathbb{T}^+_\tau z_0 + \int_0^\tau \mathbb{T}^+_{\tau-t} C^* \underbrace{C\mathbb{S}_t z_0}_{\Psi_\tau z_0} dt, \qquad \forall\, z_0 \in X.$$

- Using this (type of) Duhamel formula(s), we obtain

$$\mathbb{T}^-_\tau \mathbb{T}^+_\tau \mathrm{V}_{\mathrm{Unobs}} \subset \mathrm{V}_{\mathrm{Unobs}}, \quad \mathbb{T}^-_\tau \mathbb{T}^+_\tau \mathrm{V}_{\mathrm{Obs}} \subset \mathrm{V}_{\mathrm{Obs}}.$$

The algorithm preserves the decomposition of X !

Theorem

Denote by Π the orthogonal projection from X onto $\mathrm{V_{Obs}}$. Then the following statements hold true for all $z_0 \in X$ and $z_0^+ \in \mathrm{V_{Obs}}$:

1. For all $n \geq 1$,
$$\left\|(I - \Pi)\left(z_n^-(0) - z_0\right)\right\| = \|(I - \Pi) z_0\| .$$
2. The sequence $\left(\|\Pi\left(z_n^-(0) - z_0\right)\|\right)_{n\geq 1}$ is strictly decreasing and
$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| = \left\|z_n^-(0) - \Pi z_0\right\| \underset{n\to\infty}{\longrightarrow} 0.$$
3. There exists a constant $\alpha \in (0,1)$, independent of z_0 and z_0^+, such that for all $n \geq 1$,
$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| \leq \alpha^n \left\|z_0^+ - \Pi z_0\right\| ,$$
if and only if $\mathrm{Ran}\ \Psi_\tau^*$ is closed in X.

Example

Consider the following Scrödinger's equation

$$\left\{\begin{array}{ll} \dfrac{\partial}{\partial t} z = -\mathbf{i}\dfrac{\partial^2}{\partial x^2} z + \alpha z & \forall x \in (0,1), t \geq 0, \\ z(t,0) = z(t,1) = 0 & \forall t \geq 0, \\ z(0,x) = z_0(x) & \forall x \in (0,1), \end{array}\right.$$

with z_0 the initial state.

Example

Consider the following Scrödinger's equation

$$\begin{cases} \frac{\partial}{\partial t} z = -\mathbf{i} \frac{\partial^2}{\partial x^2} z + \alpha z & \forall x \in (0,1), t \geq 0, \\ z(t,0) = z(t,1) = 0 & \forall t \geq 0, \\ z(0,x) = z_0(x) & \forall x \in (0,1), \end{cases}$$

with z_0 the initial state.

Observation

We observe the system on $(0, 0.1)$ during a time $\tau = 0.2$, *via* one of the three following ways

$$\begin{cases} y_1(t,x) = z(t,x)|_{x \in (0,0.1)} & \forall t \in (0, 0.2), \\ y_2(t,x) = \operatorname{Re} z(t,x)|_{x \in (0,0.1)} & \forall t \in (0, 0.2), \\ y_3(t,x) = \mathbf{i} \operatorname{Im} z(t,x)|_{x \in (0,0.1)} & \forall t \in (0, 0.2). \end{cases}$$

The algorithm reads, for all $n \in \mathbb{N}$, $k = 1, 2, 3$:
Forward observers:

$$\begin{cases} \dfrac{\partial}{\partial t} z_n^+ = -\mathbf{i} \dfrac{\partial^2}{\partial x^2} z_n^+ + \alpha z_n^+ - \gamma \chi z_n^+ + \gamma y_k & \forall x \in (0,1), t \geq 0, \\ z_n^+(t,0) = z_n^+(t,1) = 0 & \forall t \geq 0, \\ z_n^+(0,x) = z_{n-1}^-(\tau, x) & \forall x \in (0,1), n \geq 1, \\ z_1^+(0,x) = 0 & \forall x \in (0,1), \end{cases}$$

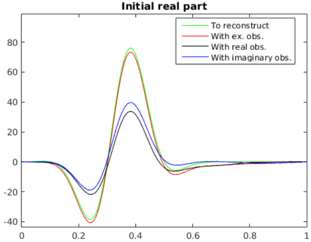
The algorithm reads, for all $n \in \mathbb{N}$, $k = 1, 2, 3$:
Forward observers:

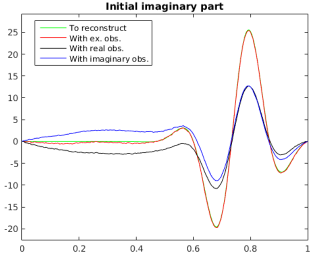
$$
\begin{cases}
\dfrac{\partial}{\partial t} z_n^+ = -\mathbf{i} \dfrac{\partial^2}{\partial x^2} z_n^+ + \alpha z_n^+ - \gamma \chi z_n^+ + \gamma y_k & \forall x \in (0,1), t \geq 0, \\
z_n^+(t,0) = z_n^+(t,1) = 0 & \forall t \geq 0, \\
z_n^+(0,x) = z_{n-1}^-(\tau, x) & \forall x \in (0,1), n \geq 1, \\
z_1^+(0,x) = 0 & \forall x \in (0,1),
\end{cases}
$$

Backward observers:

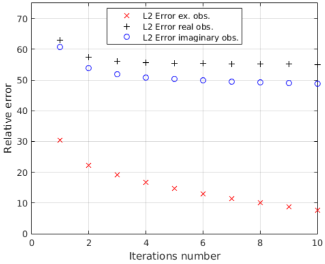
$$
\begin{cases}
\dfrac{\partial}{\partial t} z_n^- = \mathbf{i} \dfrac{\partial^2}{\partial x^2} z_n^- - \alpha z_n^- + \gamma \chi z_n^- - \gamma \text{Я}_\tau y_k & \forall x \in (0,1), t \geq 0, \\
z_n^-(t,0) = z_n^-(t,1) = 0 & \forall t \geq 0, \\
z_n^-(0,x) = z_n^+(\tau, x) & \forall x \in (0,1), n \geq 0,
\end{cases}
$$

We test with $\alpha = \pm 15$ and 0, and find in the three cases

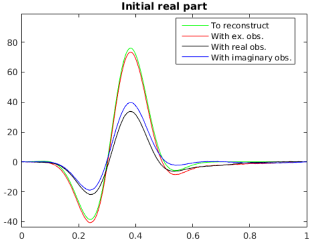


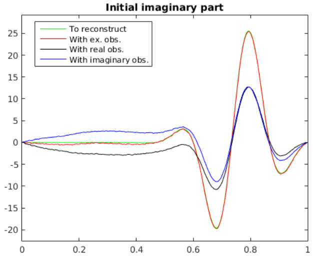


with the L^2 errors

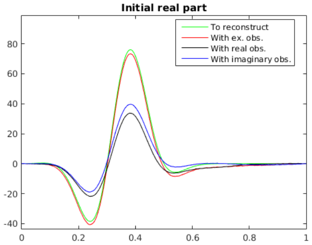


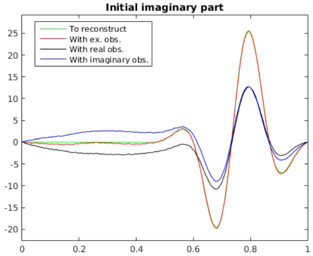
Locally distributed perturbation on $(0.75, 1)$





Locally distributed perturbation on $(0.75, 1)$





Conjecture

Let X and Y be Hilbert spaces. Assume that Σ is a well-posed linear system such that $A = \mathcal{A} + P$, for some $P \in \mathcal{L}(X)$ and skew-adjoint operator $\mathcal{A}$. Then the conclusions of the main theorem hold.

Conclusion

More ?

G. Haine
Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator
(Mathematics of Control, Signals, and Systems (MCSS), *January 2014*)

Conclusion

More ?

G. Haine
Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator
(Mathematics of Control, Signals, and Systems (MCSS), *January 2014*)

Application to thermo-acoustic tomography:

G. Haine
An observer-based approach for thermoacoustic tomography
(Mathematical Theory of Networks and Systems (MTNS – Gröningen), *July 2014*)

Conclusion

More ?

G. Haine
Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator
(Mathematics of Control, Signals, and Systems (MCSS), *January 2014*)

Application to thermo-acoustic tomography:

G. Haine
An observer-based approach for thermoacoustic tomography
(Mathematical Theory of Networks and Systems (MTNS – Gröningen), *July 2014*)

Still to be done:

- Stability of $\mathrm{V}_{\mathrm{Obs}}$ and $\mathrm{V}_{\mathrm{Unobs}}$ with noisy observation y
- More general perturbations

Thank you for your attention