An observer-based approach for thermoacoustic tomography

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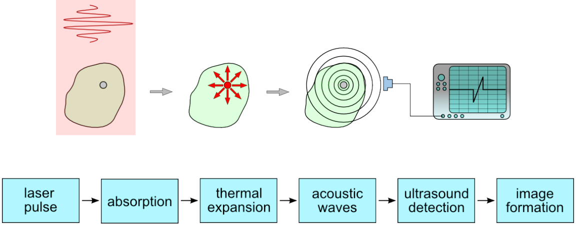
Outline

1. **Introduction**
2. The algorithm
3. Application to TAT
4. Conclusion

Introduction



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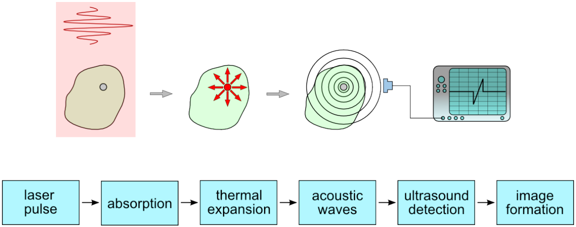

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$$\begin{cases} \dfrac{d^2 w}{dt^2}(x,t) = c^2(x)\Delta w(x,t), & \forall\, t \geq 0,\ x \in \mathbb{R}^3, \\ w(x,0) = w_0(x), & \forall\, x \in \mathbb{R}^3, \\ \dfrac{dw}{dt}(x,0) = 0, & \forall\, x \in \mathbb{R}^3, \end{cases}$$

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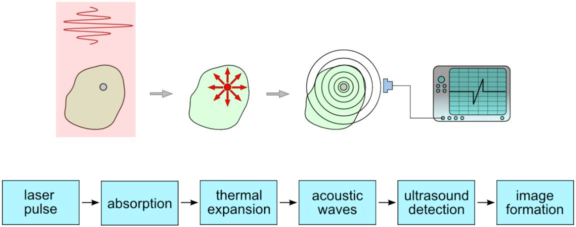

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where

- c is the **known** velocity of the wave,
- $(w_0, 0)$ is the unknown containing information on the distribution of energy absorption (which is related to cell's health).

Let

1. X, Y be two Hilbert spaces,
2. $A : \mathcal{D}(A) \subset X \to X$ a skew-adjoint operator,
3. $C \in \mathcal{L}(X, Y)$ an observation operator.

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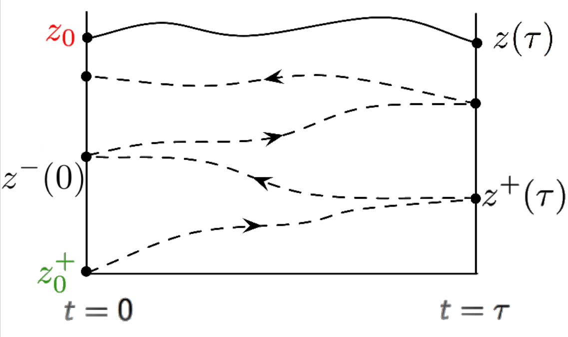
Inverse problem

Can we reconstruct z_0 from the knowledge of $y(t)$?

The algorithm

K. Ramdani, M. Tucsnak, and G. Weiss
Recovering the initial state of an infinite-dimensional system using observers (Automatica, 2010)

Intuitive representation



2 iterations, observation on $[0, \tau]$.

Some bibliography

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- **2009-2011:** Uhlmann *et al.* (*SIAM J. Imaging Sciences, Inverse Problems, ...*) use time reversal methods for solving TAT, leading to a Neumann series expansion.
  Our algorithm can lead to the same expansion (when $z_0^+ = 0$), even in ill-posed cases, and only need direct wave solver in practice.

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We construct the **forward observer**

$$\begin{cases} \dot{z}^+(t) = Az^+(t) - \gamma C^* C z^+(t) + \gamma C^* y(t), & \forall\, t \in [0, \tau], \\ z^+(0) = z_0^+ \in \mathcal{D}(A). \end{cases}$$

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$$e = z^+ - z,$$

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which is known to be exponentially stable if and only if (A, C) is exactly observable, *i.e.*

$$\exists \tau > 0, \exists k_\tau > 0, \int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(A).$$

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And from similar computations for $A^- := -A - \gamma C^* C$ as those for $A^+ := A - \gamma C^* C$:

$$\|z^-(0) - z_0\| \le M e^{-\beta\tau} \|z^+(\tau) - z(\tau)\| \le M^2 e^{-2\beta\tau} \|z_0^+ - z_0\|.$$

If the system is exactly observable in time $\tau > 0$, that is if:

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Iterating n-times the forward–backward observers with $z_n^+(0) = z_{n-1}^-(0)$ leads to

$$\|z_n^-(0) - z_0\| \leq \alpha^n \|z_0^+ - z_0\|.$$

This is the iterative algorithm of Ramdani, Tucsnak and Weiss to reconstruct z_0 from $y(t)$.

Modelling the problem

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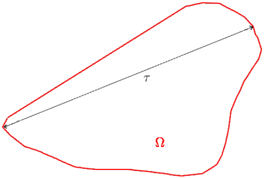

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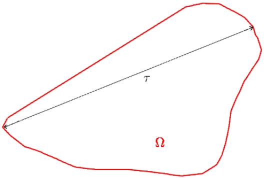

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Hence

$$y(x,t) = w(x,t), \qquad \forall\, x \in \mathcal{S}, t \in [0,\tau].$$

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$$w(x,t) = \frac{\partial}{\partial t}\left(tSw_0(x)\right), \qquad \forall\, x \in \mathbb{R}^3, t \geq 0,$$

with $Sf(x)(t) = \int_{|v|=1} f(x+tv)d\sigma(v)$.

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4. **Since we measure during** $\tau > 0$ **seconds** $\Longrightarrow$ we bound "the computation domain" by

$$\Omega_{\tau^+} = \{y \in \mathbb{R}^3 \mid |x-y| \leq \tau + \varepsilon, x \in \Omega\},$$

   for some fixed $\varepsilon > 0$.

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On Ω_{τ^+}, $w(x,t)$ is also the solution of

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$$\mathcal{D}\left(A_0^{\frac{1}{2}}\right) = H_0^1(\Omega_{\tau^+}) \to H^{\frac{1}{2}}(\partial\Omega), \qquad Y = L^2(\partial\Omega),$$

$$C_0 = \gamma_0 : \mathcal{D}\left(A_0^{\frac{1}{2}}\right) \to H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow Y.$$

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Finally, rewriting the model as a first-order system

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$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \qquad \mathcal{D}\left(A\right) = \mathcal{D}\left(A_0\right) \times \mathcal{D}\left(A_0^{\frac{1}{2}}\right), \qquad C = \begin{bmatrix} C_0 & 0 \end{bmatrix},$$

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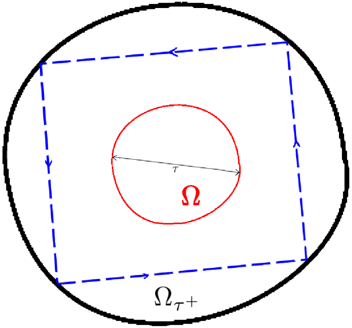

Some rays are trapped (Bardos, Lebeau, Rauch 1992).

Decomposition of X:

- Let us denote Ψ_τ the following continuous linear operator

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- We decompose $X = \mathrm{Ker}\ \Psi_\tau \oplus \left(\mathrm{Ker}\ \Psi_\tau\right)^\perp$ and define

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Note that the exact observability assumption is equivalent to Ψ_τ is bounded from below and then $\Rightarrow X = \mathrm{Ran}\ \Psi_\tau^*$.

Theorem

Denote by Π the orthogonal projection from X onto $\mathrm{V_{Obs}}$. Then the following statements hold true for all $z_0 \in X$ and $z_0^+ \in \mathrm{V_{Obs}}$:

1. For all $n \geq 1$,

$$\left\|(I - \Pi)\left(z_n^-(0) - z_0\right)\right\| = \|(I - \Pi)\, z_0\|\,.$$

2. The sequence $\left(\|\Pi\left(z_n^-(0) - z_0\right)\|\right)_{n\geq 1}$ is strictly decreasing and

$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| = \left\|z_n^-(0) - \Pi z_0\right\| \underset{n\to\infty}{\longrightarrow} 0.$$

3. There exists a constant $\alpha \in (0,1)$, independent of z_0 and z_0^+, such that for all $n \geq 1$,

$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| \leq \alpha^n \left\|z_0^+ - \Pi z_0\right\|,$$

if and only if $\mathrm{Ran}\ \Psi_\tau^*$ is closed in X.

Reconstruction algorithm

The forward observer reads

$$\begin{cases} \dot{w}_n^+(t) = -\gamma C_0^* C_0 w_n^+(t) + \widetilde{w}_n^+(t) + \gamma C_0^* y(t), & \forall\, t \in [0,\tau], \\ \dot{\widetilde{w}}_n^+(t) = -A_0 w_n^+(t), & \forall\, t \in [0,\tau], \\ w_1^+(0) = 0, & \\ \widetilde{w}_1^+(0) = 0, & \\ w_n^+(0) = w_{n-1}^-(0), & \forall\, n \geq 2, \\ \widetilde{w}_n^+(0) = \widetilde{w}_{n-1}^-(0), & \forall\, n \geq 2, \end{cases}$$

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and the backward observer is

$$\begin{cases} \dot{w}_n^-(t) = \gamma C_0^* C_0 w_n^-(t) + \widetilde{w}_n^-(t) - \gamma C_0^* y(t), & \forall\, t \in [0,\tau], \\ \dot{\widetilde{w}}_n^-(t) = -A_0 w_n^-(t)(t), & \forall\, t \in [0,\tau], \\ w_n^-(\tau) = w_n^+(\tau), & \forall\, n \geq 1, \\ \widetilde{w}_n^-(\tau) = \widetilde{w}_n^+(\tau), & \forall\, n \geq 1. \end{cases}$$

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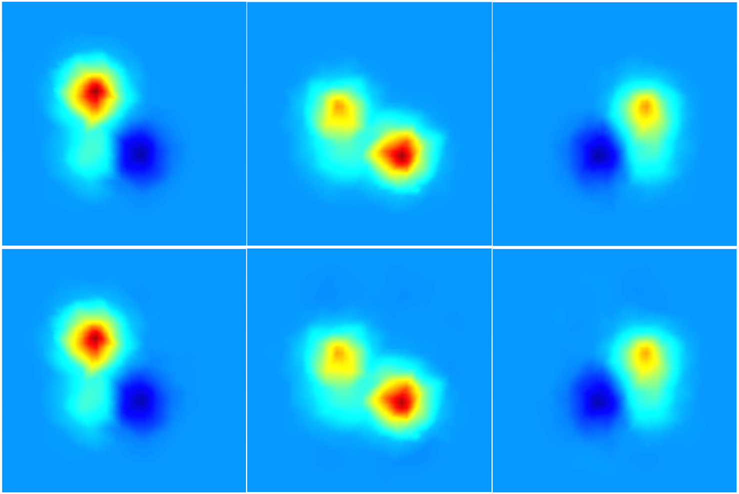
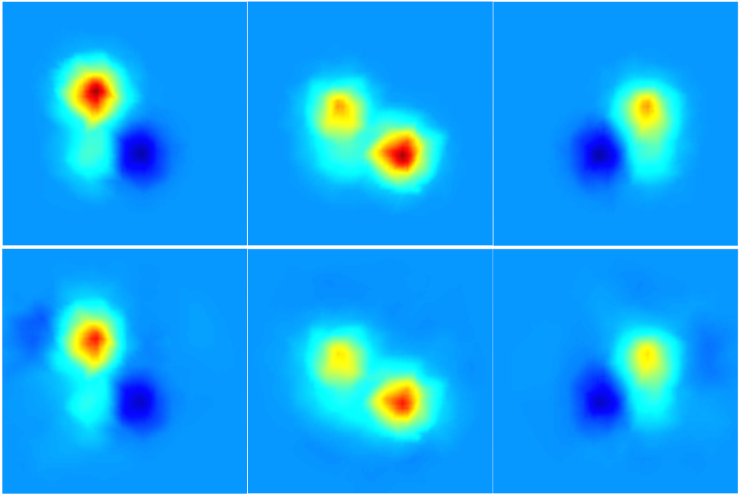
- We use this noisy observation on several configurations:
  1. We test the influence of the gain parameter γ
  2. We test ill-posed cases: lack of observation

Simulations with observation on a sphere



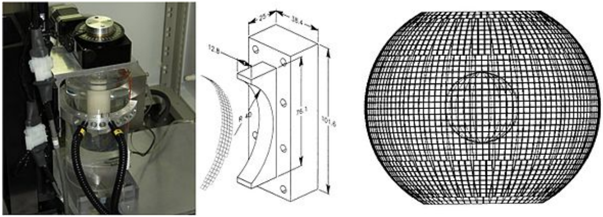
Simulations with observation on a sphere: well-posed inverse problem !

Simulations with observation on a half-sphere



Simulations with observation on a half-sphere

What about real life applications?



Small Animal Scanner – 2D Array TAT – Wikipedia (EN)

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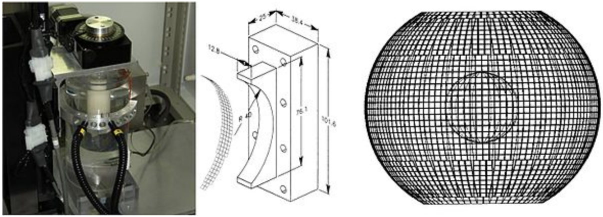

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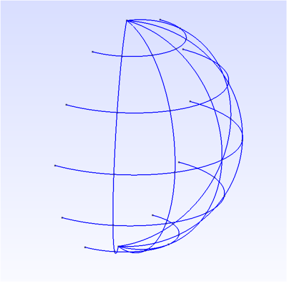
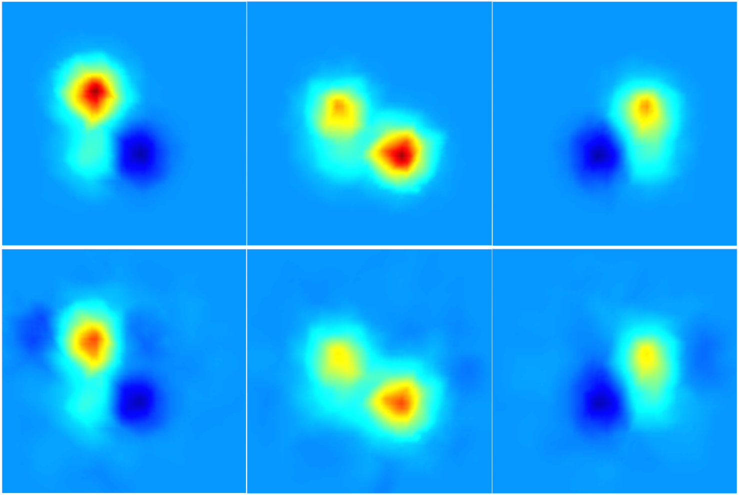
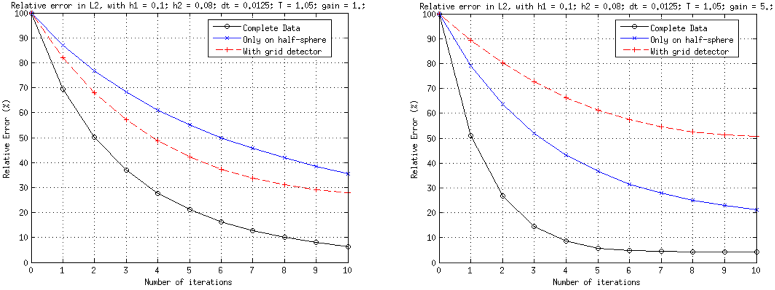


Simulations with observation on a 2D array



Simulations with observation on a 2D array on the half-sphere



Relative errors in L^2 with gain parameter $\gamma = 1$ (left) and $\gamma = 5$ (right)

Conclusion

Read more on the subject?

G. Haine
Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator
(Mathematics of Control, Signals, and Systems (MCSS), *January 2014*)

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Reconstructing initial data using observers: error analysis of the semi-discrete and fully discrete approximations
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But there is still a lot to be done:

- Stability of $\mathrm{V}_{\mathrm{Obs}}$ and $\mathrm{V}_{\mathrm{Unobs}}$ with noisy observation y
- Generalization $(A^* \neq -A)$
- Optimization of γ