# Reconstructing initial data using iterative observers for wave type systems

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- $X$ be a Hilbert space,
- $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator,

**Conservative systems**

$$\begin{cases} \dot{z}(t) = Az(t), & \forall\, t \in [0, \infty), \\ z(0) = z_0 \in \mathcal{D}(A). \end{cases}$$

For instance:

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \text{ (+ Dirichlet boundary conditions) on } \Omega \subset \mathbb{R}^n$$

and $X = H_0^1(\Omega) \times L^2(\Omega)$

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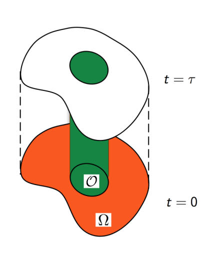
- $Y$ be another Hilbert space
- $C \in \mathcal{L}(X, Y)$
- $\tau > 0$

We observe $z$ *via* $y(t) = Cz(t)$ for all $t \in [0, \tau]$.

For instance, for the classical wave equation, let $\mathcal{O} \subset \Omega$:

$$\begin{aligned} y(t) &= \begin{bmatrix} 0 & \chi_{\mathcal{O}} \end{bmatrix} \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}, & \forall t \in [0, \tau], \\ &= \chi_{\mathcal{O}} \dot{w}(t), & \forall t \in [0, \tau]. \end{aligned}$$

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**Reconstruct the unknown $z_0$ in $X$ from the measurement $y(t)$.**

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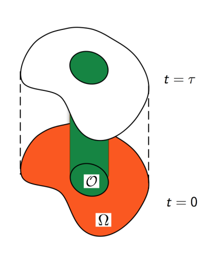


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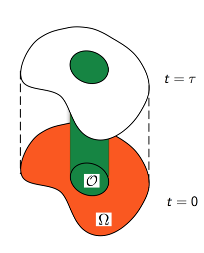


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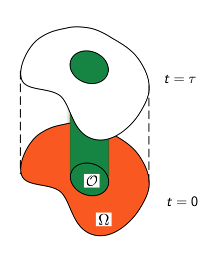


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1. The reconstruction algorithm
2. Main result
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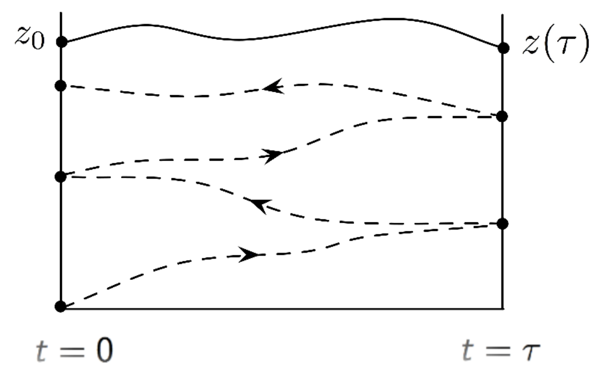
K. Ramdani, M. Tucsnak, and G. Weiss
*Recovering the initial state of an infinite-dimensional system using observers* (Automatica, 2010)

## Intuitive representation




*2 iterations, observation on* $[0, \tau]$*.*

# Some remarks

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- **2008:** Phung and Zhang (*SIAM J. Appl. Math.*) introduced the Time Reversal Focusing (TRF), for the Kirchhoff plate equation
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If we subtract the observed system

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to obtain *(remember that $y(t) = Cz(t)$)*, denoting $e = z^+ - z$,

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which is known to be exponentially stable if and only if $(A, C)$ is exactly observable, *i.e.*

$$\exists T > 0, \exists k_T > 0, \int_0^T \|y(t)\|^2 dt \geq k_T^2 \|z_0\|^2, \qquad \forall\, z_0 \in \mathcal{D}(A).$$

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We construct a similar system: the **backward observer**,

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From similar computations

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Ito, Ramdani and Tucsnak (Discrete Contin. Dyn. Syst. Ser. S, 2011) proved that

$$\alpha := M^2 e^{-2\beta\tau} < 1.$$

Iterating $n$-times the forward–backward observers with $z_n^+(0) = z_{n-1}^-(0)$ leads to

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**is not supposed to be satisfied !**

However, the algorithm doesn't need this assumption to be well-posed.

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- If it does, what is the limit of $z_n^-(0)$ and how is it related to $z_0$ ?

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- Let us denote $\Psi_\tau$ the following continuous linear operator

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*Intuitively, if $z_0$ is in Ker $\Psi_\tau$, then $y(t) \equiv 0$, and we have no information on $z_0$ !*

- We decompose $X = \text{Ker } \Psi_\tau \oplus \left(\text{Ker } \Psi_\tau\right)^\perp$ and define

$$\mathrm{V}_{\mathrm{Unobs}} = \text{Ker } \Psi_\tau, \quad \mathrm{V}_{\mathrm{Obs}} = \left(\text{Ker } \Psi_\tau\right)^\perp = \overline{\text{Ran } \Psi_\tau^*}.$$

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**Stability under the algorithm:**

Let us denote $\mathbb{T}^+$ (resp. $\mathbb{T}^-$) the semigroup generated by $A^+ := A - C^*C$ (resp. $A^- := -A - C^*C$) on $X$.

- Forward–backward observers cycle $\Rightarrow$ operator $\mathbb{T}_\tau^-\mathbb{T}_\tau^+$, *i.e.*

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## Convergence of the algorithm:

- It is obvious that the algorithm has no influence on $\mathrm{V}_{\mathrm{Unobs}}$.
- Let us denote $L = \mathbb{T}_\tau^- \mathbb{T}_\tau^+|_{\mathrm{V}_{\mathrm{Obs}}}$, we have:
  1. $$\lim_{n\to\infty} L^n z = 0, \qquad \forall\, z \in X$$
  2. $$\|L\|_{\mathcal{L}(\mathrm{V}_{\mathrm{Obs}})} < 1 \iff \text{Ran } \Psi_\tau^* \text{ is closed in } X$$

**Sketch of proof**

1. - $L$ is positive self-adjoint.
   - $L^{n+1} < L^n$ from which we get $\lim_{n\to\infty} L^n = L_\infty \in \mathcal{L}(\mathrm{V}_{\mathrm{Obs}})$.
   - $L_\infty^2 = L_\infty$ and $\|L_\infty z\| < \|z\|$ for all $z \in \mathrm{V}_{\mathrm{Obs}} \Longrightarrow$ Ran $L_\infty = \{0\}$.
2. - Duhamel formulas $\Longrightarrow \|L\|_{\mathcal{L}(\mathrm{V}_{\mathrm{Obs}})}$ in term of $\inf_{\|z\|=1, z\in \mathrm{V}_{\mathrm{Obs}}} \|\Psi_\tau z\|$.
   - Ran $\Psi_\tau^*$ closed in $X \iff \Psi_\tau$ bounded from below on $\mathrm{V}_{\mathrm{Obs}}$.

Furthermore, it is easy to prove that:

$$z_0^+ \in \mathrm{V}_{\mathrm{Obs}} \Longrightarrow z_n^-(0) \in \mathrm{V}_{\mathrm{Obs}}, \ \forall n \geq 1.$$

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## Theorem

Denote by $\Pi$ the orthogonal projection from $X$ onto $\mathrm{V_{Obs}}$. Then the following statements hold true for all $z_0 \in X$ and $z_0^+ \in \mathrm{V_{Obs}}$:

1. For all $n \geq 1$,

$$\left\|(I - \Pi)\left(z_n^-(0) - z_0\right)\right\| = \|(I - \Pi)\, z_0\| \,.$$

2. The sequence $\left(\|\Pi\left(z_n^-(0) - z_0\right)\|\right)_{n\geq 1}$ is strictly decreasing and

$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| = \left\|z_n^-(0) - \Pi z_0\right\| \underset{n\to\infty}{\longrightarrow} 0.$$

3. There exists a constant $\alpha \in (0,1)$, independent of $z_0$ and $z_0^+$, such that for all $n \geq 1$,

$$\left\|\Pi\left(z_n^-(0) - z_0\right)\right\| \leq \alpha^n \left\|z_0^+ - \Pi z_0\right\|,$$

if and only if Ran $\Psi_\tau^*$ is closed in $X$.

## Remark

Using the framework of well-posed linear systems, we obtain the same result for some unbounded observation operator $C \in \mathcal{L}(\mathcal{D}(A), Y)$.

## Example

Let

- $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$
- $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$

Consider the following wave system

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = 0, & \forall x \in \Omega, t > 0, \\ w(x,t) = 0, & \forall x \in \Gamma_0, t > 0, \\ w(x,t) = u(x,t), & \forall x \in \Gamma_1, t > 0, \\ w(x,0) = w_0(x), \ \dot{w}(x,0) = w_1(x), & \forall x \in \Omega, \end{cases}$$

with $u$ the control, and $(w_0, w_1)$ the initial state.

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## Observation

Let $\nu$ be the unit normal vector of $\Gamma_1$ pointing towards the exterior of $\Omega$, we observe the system *via*

$$y(x,t) = -\frac{\partial(-\Delta)^{-1}\dot{w}(x,t)}{\partial\nu}, \quad \forall x \in \Gamma_1, t > 0.$$

- Guo and Zhang (SIAM J. Control Optim., 2005) $\Rightarrow$ well-posed linear system.
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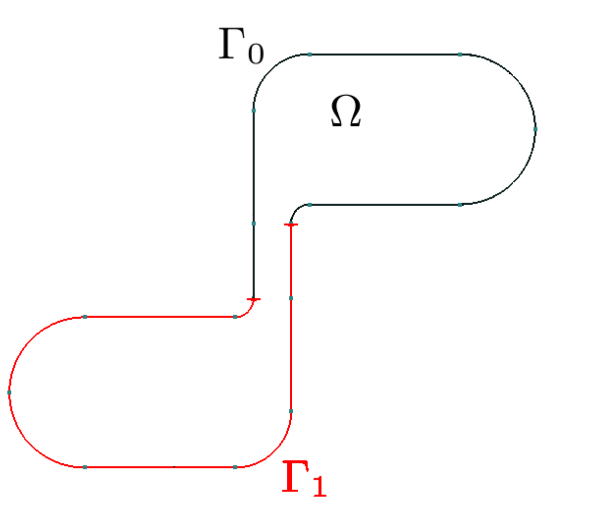

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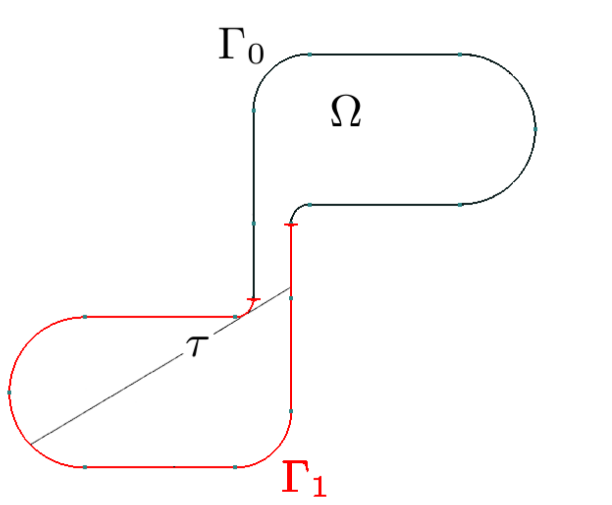

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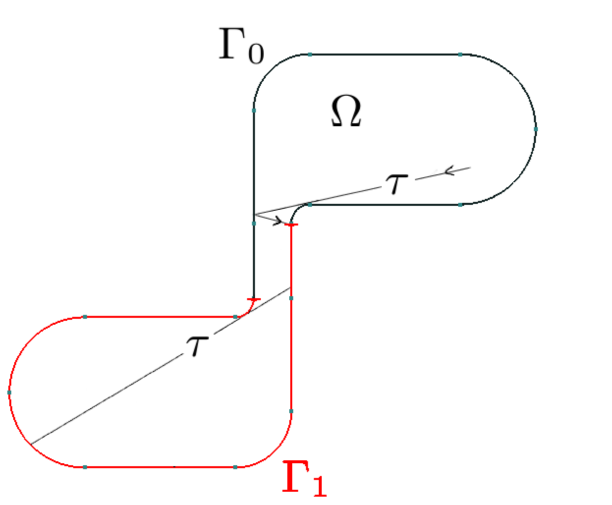

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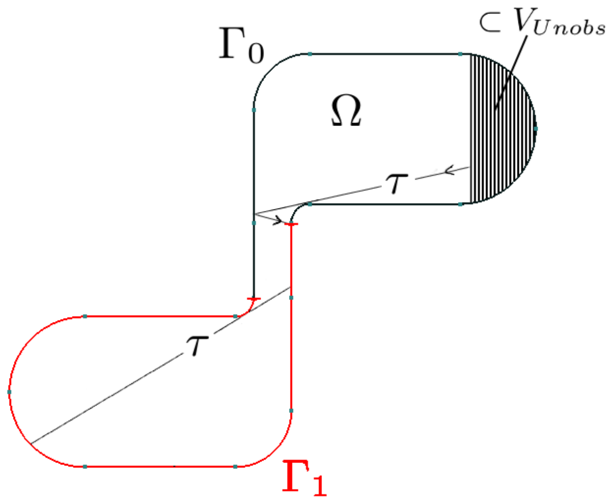

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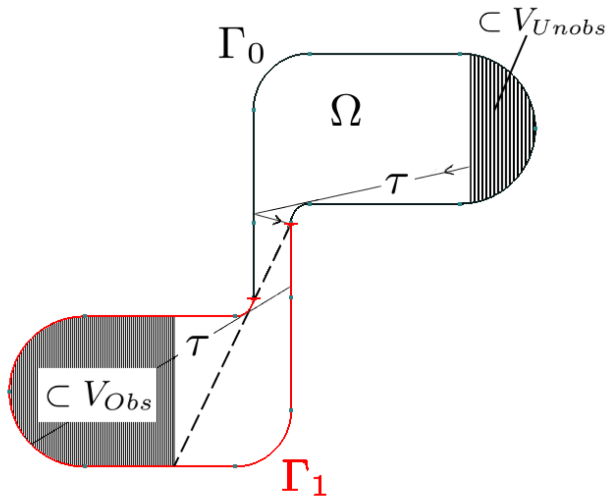

Choosing a suitable initial data

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To perform the test, we use

- Gmsh: a 3D finite element grid generator
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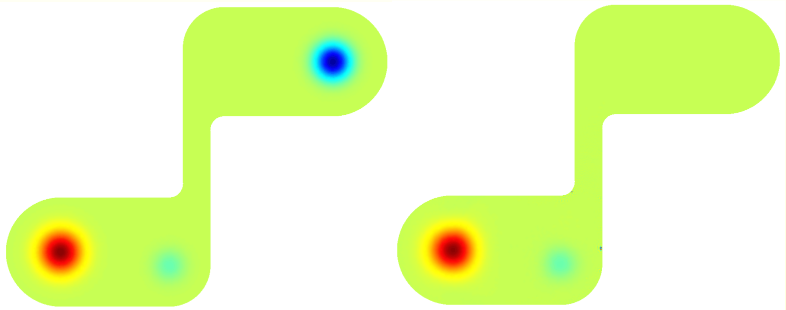

*The initial position and its reconstruction after 3 iterations*

$\Rightarrow$ 6% of relative error in $L^2(\Omega)$ on the "observable part".

**Work-in-progress:**

Application to thermo-acoustic tomography (simulations in progress)

**Still to be done:**

- Stability of $\mathrm{V}_{\mathrm{Obs}}$ and $\mathrm{V}_{\mathrm{Unobs}}$ with noisy observation $y$
- Generalization $(A^* \neq -A)$

# Thanks for your attention !

G. Haine
*Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint operator*
(Mathematics of Control, Signals, and Systems (MCSS), *In Revision*)