Reconstructing initial data using iterative observers for wave type systems. A numerical analysis.

Ghislain Haine, Karim Ramdani

Institut Élie Cartan de Nancy (IECN) – INRIA – CNRS
FRANCE

WAVES 2011 - Simon Fraser University - July, 25–29

Let

- H be a Hilbert space,
- $A_0 : \mathcal{D}(A_0) \to H$ be a positive self-adjoint operator,

Wave-type system

$$\begin{cases} \ddot{w}(t) + A_0 w(t) = 0, & \forall\, t \in [0, \infty), \\ w(0) = w_0 \in \mathcal{D}(A_0), \\ \dot{w}(0) = w_1 \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right). \end{cases}$$

For instance, if $A_0 = -\Delta$ with some suitable boundary conditions on a domain $\Omega \subset \mathbb{R}^n$ and $H = L^2(\Omega)$, we get the classical wave equation.

Let

- H be a Hilbert space,
- $A_0 : \mathcal{D}(A_0) \to H$ be a positive self-adjoint operator,

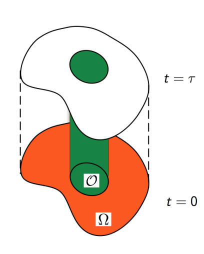
Wave-type system

$$\begin{cases} \ddot{w}(t) + A_0 w(t) = 0, & \forall\ t \in [0, \infty), \\ w(0) = w_0 \in \mathcal{D}(A_0), \\ \dot{w}(0) = w_1 \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right). \end{cases}$$

For instance, if $A_0 = -\Delta$ with some suitable boundary conditions on a domain $\Omega \subset \mathbb{R}^n$ and $H = L^2(\Omega)$, we get the classical wave equation.

We observe the velocity $\dot{w}$ of this system on a non-empty subdomain $\mathcal{O}$, over a time interval $[0, \tau]$, leading to the measurement

$$y(t) = \chi_{|\mathcal{O}}\dot{w}(t).$$



Observation on $\mathcal{O} \times [0, \tau]$.

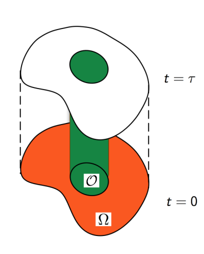
Our problem

Reconstruct the unknown (w_0, w_1) in $\mathcal{D}\left(A_0^{\frac{1}{2}}\right) \times H$ from the measurement $y(t)$.

A similar problem arises for instance in Thermo-Acoustic Tomography.

We observe the velocity $\dot{w}$ of this system on a non-empty subdomain $\mathcal{O}$, over a time interval $[0, \tau]$, leading to the measurement

$$y(t) = \chi_{|\mathcal{O}} \dot{w}(t).$$



Observation on $\mathcal{O} \times [0, \tau]$.

Our problem

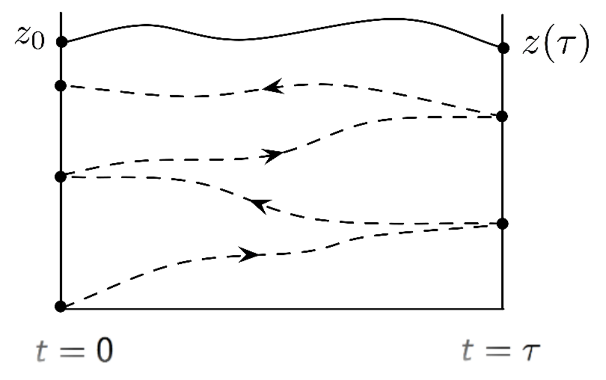
Reconstruct the unknown (w_0, w_1) **in** $\mathcal{D}\left(A_0^{\frac{1}{2}}\right) \times H$ **from the measurement** $y(t)$**.**

A similar problem arises for instance in Thermo-Acoustic Tomography.

1. The reconstruction algorithm
2. Main result
3. Numerical study
4. Conclusion

K. Ramdani, M. Tucsnak, and G. Weiss
Recovering the initial state of an infinite-dimensional system using observers (Automatica, 2010)

Intuitive representation



2 iterations, observation on $[0, \tau]$.

We construct the **forward observer**

$$\begin{cases} \ddot{w}^+(t) + A_0 w^+(t) + \gamma \chi_{|\mathcal{O}} \dot{w}^+(t) = \gamma y(t), & \forall\, t \in [0,\tau], \\ w^+(0) = 0, \\ \dot{w}^+(0) = 0, \end{cases}$$

$$-\begin{cases} \ddot{w}(t) + A_0 w(t) = 0, & \forall\, t \in [0,\tau], \\ w(0) = w_0, \\ \dot{w}(0) = w_1, \end{cases}$$

remember that $y(t) = \chi_{|\mathcal{O}} \dot{w}(t)$

$$= \begin{cases} \ddot{e}(t) + A_0 e(t) + \gamma \chi_{|\mathcal{O}} \dot{e}(t) = 0, & \forall\, t \in [0,\tau], \\ e(0) = -w_0, \\ \dot{e}(0) = -w_1, \end{cases}$$

which is known to be exponentially stable for suitable $\mathcal{O}$ and τ (for instance verifying Geometric Optic Condition of Bardos, Lebeau and Rauch (1992) in the classical wave case) and all $\gamma > 0$.

We construct the **forward observer**

$$\left\{\begin{array}{ll} \ddot{w}^+(t) + A_0 w^+(t) + \gamma \chi_{|\mathcal{O}} \dot{w}^+(t) = \gamma y(t), & \forall\ t \in [0,\tau], \\ w^+(0) = 0, & \\ \dot{w}^+(0) = 0, & \end{array}\right.$$

$$-\left\{\begin{array}{ll} \ddot{w}(t) + A_0 w(t) = 0, & \forall\ t \in [0,\tau], \\ w(0) = w_0, & \\ \dot{w}(0) = w_1, & \end{array}\right.$$

remember that $y(t) = \chi_{|\mathcal{O}} \dot{w}(t)$

$$=\left\{\begin{array}{ll} \ddot{e}(t) + A_0 e(t) + \gamma \chi_{|\mathcal{O}} \dot{e}(t) = 0, & \forall\ t \in [0,\tau], \\ e(0) = -w_0, & \\ \dot{e}(0) = -w_1, & \end{array}\right.$$

which is known to be exponentially stable for suitable $\mathcal{O}$ and τ (for instance verifying Geometric Optic Condition of Bardos, Lebeau and Rauch (1992) in the classical wave case) and all $\gamma > 0$.

The exponential stability gives the existence of two constants $M > 0$ and $\beta > 0$ such that

$$\|\dot{w}^+(\tau) - \dot{w}(\tau)\| + \|w^+(\tau) - w(\tau)\|_{\frac{1}{2}} \leq Me^{-\beta\tau}\left(\|w_1\| + \|w_0\|_{\frac{1}{2}}\right).$$

We construct a similar system, called **backward observer**.

$$\begin{cases} \ddot{w}^-(t) + A_0 w^-(t) - \gamma\chi_{|\mathcal{O}}\dot{w}^-(t) = -\gamma y(t), & \forall\, t \in [0,\tau], \\ w^-(\tau) = w^+(\tau), \\ \dot{w}^-(\tau) = w^+(\tau), \end{cases}$$

Then, we easily get that

$$\begin{aligned} \|\dot{w}^-(0) - w_1\| + \|w^-(0) - w_0\|_{\frac{1}{2}} & \\ \leq Me^{-\beta\tau}\left(\|\dot{w}^+(\tau) - \dot{w}(\tau)\| + \|w^+(\tau) - w(\tau)\|_{\frac{1}{2}}\right), & \\ \leq Me^{-2\beta\tau}\left(\|w_1\| + \|w_0\|_{\frac{1}{2}}\right). & \end{aligned}$$

The exponential stability gives the existence of two constants $M > 0$ and $\beta > 0$ such that

$$\|\dot{w}^+(\tau) - \dot{w}(\tau)\| + \|w^+(\tau) - w(\tau)\|_{\frac{1}{2}} \leq M e^{-\beta\tau} \left(\|w_1\| + \|w_0\|_{\frac{1}{2}} \right).$$

We construct a similar system, called **backward observer**.

$$\begin{cases} \ddot{w}^-(t) + A_0 w^-(t) - \gamma \chi_{|\mathcal{O}} \dot{w}^-(t) = -\gamma y(t), & \forall\, t \in [0, \tau], \\ w^-(\tau) = w^+(\tau), \\ \dot{w}^-(\tau) = w^+(\tau), \end{cases}$$

Then, we easily get that

$$\begin{aligned} \|\dot{w}^-(0) - w_1\| + \|w^-(0) - w_0\|_{\frac{1}{2}} & \\ \leq M e^{-\beta\tau} \left(\|\dot{w}^+(\tau) - \dot{w}(\tau)\| + \|w^+(\tau) - w(\tau)\|_{\frac{1}{2}} \right), & \\ \leq M e^{-2\beta\tau} \left(\|w_1\| + \|w_0\|_{\frac{1}{2}} \right). & \end{aligned}$$

The exponential stability gives the existence of two constants $M > 0$ and $\beta > 0$ such that

$$\|\dot{w}^+(\tau) - \dot{w}(\tau)\| + \|w^+(\tau) - w(\tau)\|_{\frac{1}{2}} \leq Me^{-\beta\tau}\left(\|w_1\| + \|w_0\|_{\frac{1}{2}}\right).$$

We construct a similar system, called **backward observer**.

$$\left\{\begin{array}{ll} \ddot{w}^-(t) + A_0 w^-(t) - \gamma\chi_{|\mathcal{O}}\dot{w}^-(t) = -\gamma y(t), & \forall\, t \in [0,\tau], \\ w^-(\tau) = w^+(\tau), & \\ \dot{w}^-(\tau) = w^+(\tau), & \end{array}\right.$$

Then, we easily get that

$$\begin{aligned} &\|\dot{w}^-(0) - w_1\| + \|w^-(0) - w_0\|_{\frac{1}{2}} \\ &\qquad \leq Me^{-\beta\tau}\left(\|\dot{w}^+(\tau) - \dot{w}(\tau)\| + \|w^+(\tau) - w(\tau)\|_{\frac{1}{2}}\right), \\ &\qquad\qquad \leq Me^{-2\beta\tau}\left(\|w_1\| + \|w_0\|_{\frac{1}{2}}\right). \end{aligned}$$

Ito, Ramdani and Tucsnak (2011) showed that $\alpha := Me^{-2\beta\tau} < 1$. Thus the reconstruction of (w_0, w_1) can be achieved by iterating these two systems by taking $(w^+(0), \dot{w}^+(0)) = (w^-(0), \dot{w}^-(0))$. This leads to the algorithm

$$\begin{cases} \ddot{w}_n^+(t) + A_0 w_n^+(t) + \gamma \chi_{|\mathcal{O}} \dot{w}_n^+(t) = \gamma y(t), & \forall\, t \in [0, \tau], \\ w_n^+(0) = w_{n-1}^-(0), \quad n \geq 1, \quad w_0^+(0) = 0, \\ \dot{w}_n^+(0) = \dot{w}_{n-1}^-(0), \quad n \geq 1, \quad \dot{w}_0^+(0) = 0, \end{cases}$$

$$\begin{cases} \ddot{w}_n^-(t) + A_0 w_n^-(t) - \gamma \chi_{|\mathcal{O}} \dot{w}_n^-(t) = -\gamma y(t), & \forall\, t \in [0, \tau], \\ w_n^-(\tau) = w_n^+(\tau), \quad n \geq 0, \\ \dot{w}_n^-(\tau) = w_n^+(\tau), \quad n \geq 0. \end{cases}$$

For all $N \geq 1$, we then have

$$\|\dot{w}_N^-(0) - w_1\| + \|w_N^-(0) - w_0\|_{\frac{1}{2}} \leq \alpha^N \left(\|w_1\| + \|w_0\|_{\frac{1}{2}} \right).$$

1. The reconstruction algorithm
2. Main result
3. Numerical study
4. Conclusion

Question

Can we obtain an error estimate for the reconstruction algorithm in its fully discretized version ?

We need some regularity assumptions

- $w_0 \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right)$, $w_1 \in \mathcal{D}(A_0)$,
- $\chi_{|\mathcal{O}}$ will be replaced by a smooth cut-off function.

In the sequel, let

- h be the mesh size of the space discretization,
- Δt be the time step of the time discretization,
- $N_{h,\Delta t}$ be the (finite) number of iterations.

Question

Can we obtain an error estimate for the reconstruction algorithm in its fully discretized version ?

We need some regularity assumptions

- $w_0 \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right)$, $w_1 \in \mathcal{D}(A_0)$,
- $\chi_{|\mathcal{O}}$ will be replaced by a smooth cut-off function.

In the sequel, let

- h be the mesh size of the space discretization,
- Δt be the time step of the time discretization,
- $N_{h,\Delta t}$ be the (finite) number of iterations.

Question

Can we obtain an error estimate for the reconstruction algorithm in its fully discretized version ?

We need some regularity assumptions

- $w_0 \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right)$, $w_1 \in \mathcal{D}(A_0)$,
- $\chi_{|\mathcal{O}}$ will be replaced by a smooth cut-off function.

In the sequel, let

- h be the mesh size of the space discretization,
- Δt be the time step of the time discretization,
- $N_{h,\Delta t}$ be the (finite) number of iterations.

- **Galerkin space discretization**
$(H_h)_{h>0}$ is a family of finite-dimensional subspaces of $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ such that there exist $M > 0$, $\theta > 0$ and $h^* > 0$ such that

$$\|\pi_h \varphi - \varphi\| \leq M h^\theta \|\varphi\|_{\frac{1}{2}}, \qquad \forall\, \varphi \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right),\ h \in (0, h^*).$$

- **Implicit finite difference discretization in time**
$[0, \tau]$ is splitting with a time step $\Delta t > 0$: $t_k = k\Delta t$, with $0 \leq k \leq K$. We approximate the first and second derivative at time t_k of a function f by

$$f'(t_k) \simeq \frac{f(t_k) - f(t_{k-1})}{\Delta t},$$

$$f''(t_k) \simeq \frac{f(t_k) - 2f(t_{k-1}) + f(t_{k-2})}{\Delta t^2}.$$

- **Galerkin space discretization**
  $(H_h)_{h>0}$ is a family of finite-dimensional subspaces of $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ such that there exist $M > 0$, $\theta > 0$ and $h^* > 0$ such that

$$\|\pi_h \varphi - \varphi\| \leq M h^\theta \|\varphi\|_{\frac{1}{2}}, \qquad \forall\, \varphi \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right),\ h \in (0, h^*).$$

- **Implicit finite difference discretization in time**
  $[0, \tau]$ is splitting with a time step $\Delta t > 0$: $t_k = k\Delta t$, with $0 \leq k \leq K$. We approximate the first and second derivative at time t_k of a function f by

$$f'(t_k) \simeq \frac{f(t_k) - f(t_{k-1})}{\Delta t},$$

$$f''(t_k) \simeq \frac{f(t_k) - 2f(t_{k-1}) + f(t_{k-2})}{\Delta t^2}.$$

Theorem

Let $(w_0, w_1) \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$ and denote $(w_{0,h,\Delta t}, w_{1,h,\Delta t})$ the numerical reconstruction of (w_0, w_1).

Taking $N_{h,\Delta t} = \frac{\ln(h^\theta + \Delta t)}{\ln \alpha}$ iterations, there exist $M_\tau > 0$, $h^* > 0$ and $\Delta t^* > 0$ such that for all $h \in (0, h^*)$ and $\Delta t \in (0, \Delta t^*)$

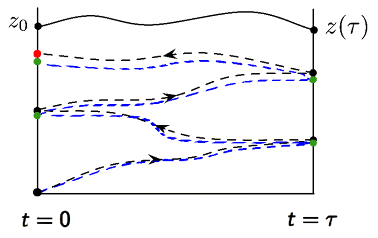
$$\left\| w_0 - w_{0,h,\Delta t} \right\|_{\frac{1}{2}} + \left\| w_1 - w_{1,h,\Delta t} \right\| \leq M_\tau \Bigg[(h^\theta + \Delta t) \ln^2(h^\theta + \Delta t) \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 \right) + \left| \ln(h^\theta + \Delta t) \right| \Delta t \sum_{\ell=0}^{K} \left\| y(t_\ell) - y_h^\ell \right\| \Bigg].$$

To prove this, we split the error

$$\left\| w_0 - w_{0,h,\Delta t} \right\|_{\frac{1}{2}} + \left\| w_1 - w_{1,h,\Delta t} \right\|$$

into three parts, taking into account the fact that

1. we stop the iteration,
2. we discretize the observers,
3. we take approximation as initial and final data.



3 error types.

Consider the 1D wave equation with unit speed on $(0,1)$ and observe the velocity of the sub-interval $(0,0.1)$ (in red) during 2 seconds.

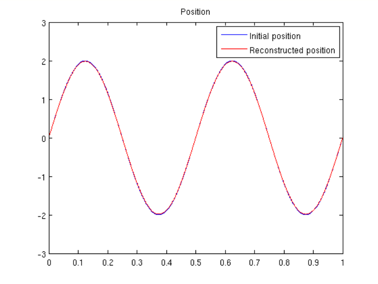
We code the algorithm presented above on Matlab, and focus our attention on three aspects

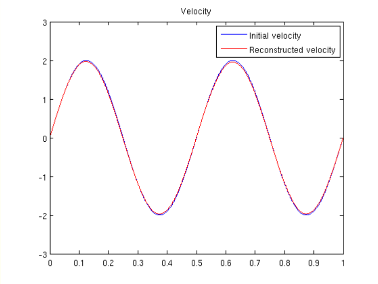
- Quality of the reconstruction
- Influence of the gain coefficient (parameter γ)
- Robustness to noise

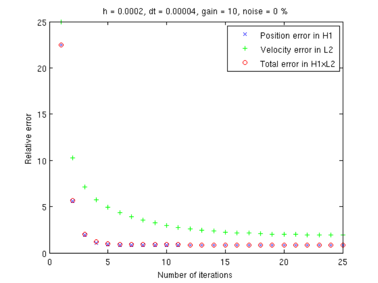
Consider the 1D wave equation with unit speed on $(0,1)$ and observe the velocity of the sub-interval $(0,0.1)$ (in red) during 2 seconds.

We code the algorithm presented above on Matlab, and focus our attention on three aspects

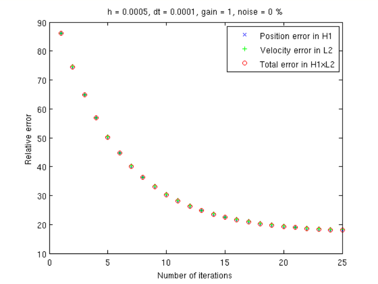
- Quality of the reconstruction
- Influence of the gain coefficient (parameter γ)
- Robustness to noise

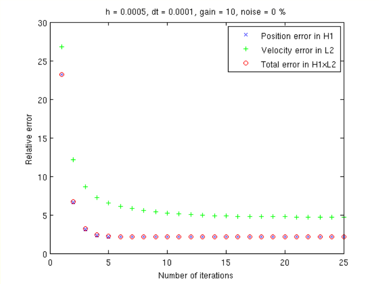


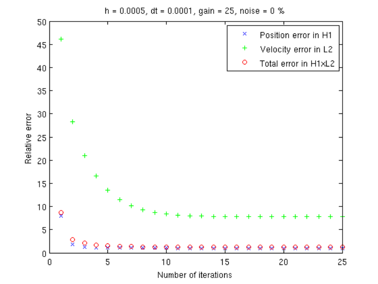




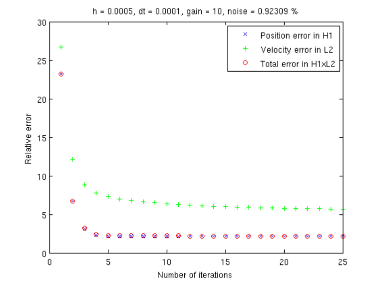
$$\gamma = 10,\ h = 2.10^{-4}\ \textit{and}\ \Delta t = 4.10^{-5}$$

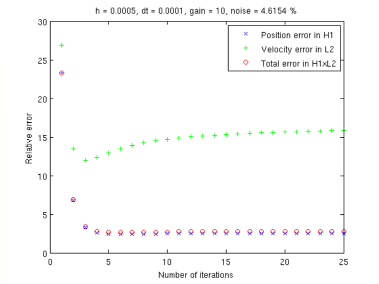


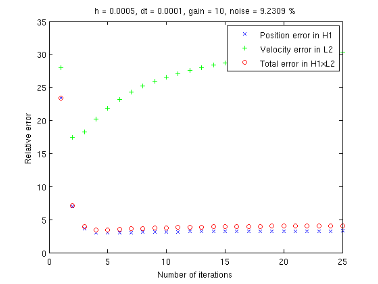




Influence of the gain coefficient : $\gamma = 1, 10, 25$







Robustness to noise : 1%, 5%, 10%

- **Maxwell's equations**

$$\begin{cases} \varepsilon \dot{E} - \mathrm{rot}\, H = 0, & \Omega, \\ \mu \dot{H} + \mathrm{rot}\, E = 0, & \Omega, \\ \mathrm{div}\, E = 0,\ \mathrm{div}\, H = 0, & \Omega, \\ E \wedge \nu = 0,\ H \cdot \nu = 0, & \partial\Omega, \\ E(.,0) = E_0, & \Omega, \\ H(.,0) = H_0, & \Omega, \end{cases} \qquad y = \chi E.$$

We are able to reconstruct (E_0, H_0) from y.

- **Source identification**

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = \lambda(t) j(x), & \Omega \times \mathbb{R}^+, \\ w(x,t) = 0, & \partial\Omega \times \mathbb{R}^+, \\ w(x,0) = 0, & \Omega, \\ \dot{w}(x,0) = 0, & \Omega, \end{cases} \qquad y(t) = \chi \dot{w}(t).$$

We are able to reconstruct j from y.

- **Stability under perturbations**

$$\begin{cases} \ddot{w} - \Delta w = 0, & \Omega, \\ w = 0, & \partial\Omega, \\ w(.,0) = w_0, & \Omega, \\ \dot{w}(.,0) = w_1, & \Omega. \end{cases}$$

$$\Downarrow ?$$

$$\begin{cases} \ddot{w} - \Delta w + w^3 = 0, & \Omega, \\ w = 0, & \partial\Omega, \\ w(.,0) = w_0, & \Omega, \\ \dot{w}(.,0) = w_1, & \Omega. \end{cases}$$

Thanks for your attention !

G. Haine and K. Ramdani
Reconstructing initial data using observers : error analysis of the semi-discrete and fully discrete approximations
(Numerische Mathematik, *In Revision*)