Data-driven structured identification of 2D Maxwell's equation as a port-Hamiltonian system

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***Abstract*—The objective of this work is to provide a systematic procedure for Reduced Order Modelling of the 2D Maxwell's equations with collocated boundary control and observation, on the basis of frequency domain data obtained thanks to a Mixed Finite Element Method: both the High Fidelity and the Low Fidelity Models share the so-called port-Hamiltonian structure (pHs) of the original PDE, which is preserved through the different steps. The efficient reduction technique relies on the Loewner framework, it is based on Benner et al. contribution [1], which has been recently adapted to handle non-strictly passive model, and numerical issues observed when dealing with complex configurations.** [1]

I. INTRODUCTION

An efficient numerical representation of Maxwell's equations on a 2D domain with boundary-*collocated* actuators and sensors is given as a port-Hamiltonian system (pHs), [2]. Recasting the PDE as a pHs [3], and applying a structure-preserving method (the Partitioned Finite Element Method (PFEM) [4]) leads to a high-fidelity full-order model (FOM), which mimics the power balance with Poynting vector. However, such a high dimension is a limiting factor for simulation, optimisation, analysis and control. The goal is to use the Loewner framework, [5], [1] to approximate the FOM with a smaller and simpler system with the *same pHs structure* and similar response characteristics as the original, the low-complexity model, a reduced-order model (ROM). The worked out example of a transverse electric (TE) field in a waveguide is provided.

Outline. The paper is organized as follows: writing 2D Maxwell's equations as an infinite-dimensional port-Hamiltonian system is recalled in § II; making use of the Mixed Finite Element Method (MFEM) to pass from the infinite-dimensional pHs to a finite-dimensional port-Hamiltonian system of high dimension is detailed in § III; then, § IV explains how to obtain a low-dimensional pHs, using the Loewner framework.

II. PDE MODEL

As main example, we shall concentrate on the 2D Maxwell's equation on a 2D waveguide for sake of simplicity. Let us consider a *bidimensional* waveguide $\Omega = \prod_{i=1}^{2}(0, L_i) \subset \mathbb{R}^2$ with $L_1 = 7$ and $L_2 = 0.1$, sketched

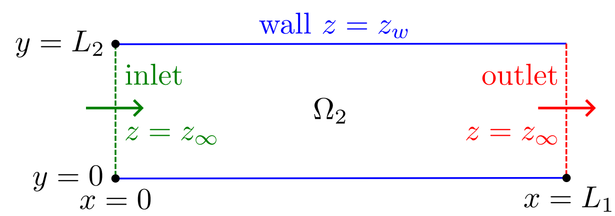

Figure 1. Waveguide sketch (arrows indicate propagation direction)

in Figure 1. Maxwell formulations yield a *transverse electric* (TE) field $(\boldsymbol{E}, H, y_\partial, u_\partial)$, where $\boldsymbol{E}$ is a 2D vector field, and H is a 2D scalar field.

We consider the macroscopic Maxwell equations

$$\begin{array}{ll} \text{(a) } \partial_t \boldsymbol{D} - \nabla \times \boldsymbol{H} = -\boldsymbol{j}, & \text{(b) } \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0, \\ \text{(c) } \nabla \cdot \boldsymbol{D} = \rho, & \text{(d) } \nabla \cdot \boldsymbol{B} = 0. \end{array} \quad (1)$$

In the 2D case, (1) hold with the definitions, see e.g. [6, § 9.2]:

$$\nabla \times H = \mathbf{grad}^\perp(H_z) := \begin{bmatrix} \partial_y H_z \\ -\partial_x H_z \end{bmatrix} = \Theta\, \mathbf{grad}(H_z),$$
$$\nabla \times \boldsymbol{E} = \mathrm{curl}_{2D}(\boldsymbol{E}) := \partial_x E_y - \partial_y E_x = \mathrm{div}_{2D}(\Theta^\top \boldsymbol{E}),$$

where Θ is the rotation of angle $\frac{\pi}{2}$. Let us define $\pi_t(\boldsymbol{E}) := (\boldsymbol{E} \cdot \boldsymbol{t})\, \boldsymbol{t}$, and $\gamma_t(H_z) := H_z\, \boldsymbol{t}$, where $\boldsymbol{t}$ is the unit tangent vector to $\partial\Omega$, defined by $\boldsymbol{t} := \Theta\, \boldsymbol{n}$, with $\boldsymbol{n}$ the outward unit normal vector. For smooth fields, the transverse differential operators satisfy the following Green formula:

$$\left(\mathbf{grad}^\perp H_z, \boldsymbol{E}\right)_\Omega = (H_z, \mathrm{curl}_{2D}\, \boldsymbol{E})_\Omega + (\pi_t(\boldsymbol{E}), \gamma_t\,(H_z))_{\partial\Omega}\,.$$

This shows that $\mathbf{grad}^\perp$ and curl_{2D} are formal adjoints. Moreover the boundary term is exactly related to the opposite of the normal trace of the Poynting vector, since: $\boldsymbol{\Pi} := \boldsymbol{E} \wedge \boldsymbol{H} = \begin{bmatrix} E_y\, H_z & -E_x\, H_z & 0 \end{bmatrix}^\top$, and $\boldsymbol{\Pi} \cdot \boldsymbol{n} = H_z\,(E_y n_x - E_x n_y)$.

Taking as Hamiltonian the total electromagnetic energy

$$\mathcal{E}(t) := \frac{1}{2} \int_\Omega \left(\boldsymbol{E}(x,y) \cdot \boldsymbol{D}(x,y) + H_z(x,y)\, B_z(x,y)\right)\, dx\, dy\,,$$

and using as boundary collocated control and observations: $u_\partial := \boldsymbol{E} \cdot \boldsymbol{t}$ and $y_\partial := -H_z$, we get the following power balance of a passive dynamical system:

$$\frac{d}{dt}\mathcal{E}(t) = -(\sigma\, \boldsymbol{E}, \boldsymbol{E})_\Omega + (u_\partial,\, y_\partial)_{\partial\Omega} \leq \int_{\partial\Omega} u_\partial\, y_\partial\, ds\,. \quad (2)$$

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III. HIGH-FIDELITY MODEL

In order to spatially discretize the system, PFEM is used, see [4], which allows discretizing pHs in a *structure-preserving* way. The method boils down to three steps:

1) write a weak formulation of the problem,
2) select a *partition* of the variables, use Stokes theorem to perform an integration by parts which makes appear the useful control in the boundary term,
3) choose a set of finite element families for the state and control variables.

Following [7] and [8], and using specific finite elements [9], a finite-dimensional dynamical system is found to be:

$$\begin{pmatrix} M_\epsilon \, \underline{\dot{E}} \\ M_\mu \, \underline{\dot{H}} \end{pmatrix} = \begin{pmatrix} -\langle\sigma\rangle & C \\ -C^\top & 0 \end{pmatrix} \begin{pmatrix} \underline{E} \\ \underline{H} \end{pmatrix} - \begin{pmatrix} 0 \\ T \end{pmatrix} \underline{u}_\partial \,, \qquad (3)$$

with conjugated output: $M_\partial \, \underline{y}_\partial = -T^\top \, \underline{H}$. Then, the power balance reads, at the discrete level:

$$\frac{d}{dt}\mathcal{E}^d(t) = -\underline{E}(t)^\top \langle\sigma\rangle \, \underline{E}(t) + \underline{y}_\partial(t)^\top M_\partial \, \underline{u}_\partial(t) \leq (\underline{u}_\partial(t), \underline{y}_\partial(t)) \,.$$

meaning that the open dynamical system is passive.

Numerical Results. The effective application of the PFEM has been done using continuous Lagrange finite elements of order 2 for the magnetic field H, while the vectorial electric field E and the boundary variables are both approximated with discontinuous polynomials of order 1 on each component. Two mesh size parameters have been chosen, leading respectively to about N = 6,000 and 50,000 degrees of freedom.

IV. LOW-FIDELITY MODEL

The Loewner framework (LF) employed in this work is a *data-driven* model identification and reduction technique that was originally introduced in [5]. Using only (complex) frequency-domain data obtained from a simulator or an experimental setup, the LF directly constructs surrogate models directly with a low computational effort. Its extension to pHs is proposed in [1]. It has been tested on 1D models in [10]. An application to the 2D wave PDE was made in [11], where the *non-strict passivity* could be tackled thanks to ad-hoc modifications of the original algorithm.

Numerical results. For the illustration, we consider a setup where all inputs and outputs are merged, thus leading to a SISO model. Figure 2 presents the Bode phase response of the original model (*data*, used for the LF) and the identified *pH-Loewner* both the full interpolant ($n = 204$) and a reduced one ($n = 59$). In both cases, models are passive and embed the expected pH-structure. In addition, Figure 3 shows the time response of the original and identified models in response to a left input on the wave-guide. In both cases, the response is well reproduced by the pH-ROM.

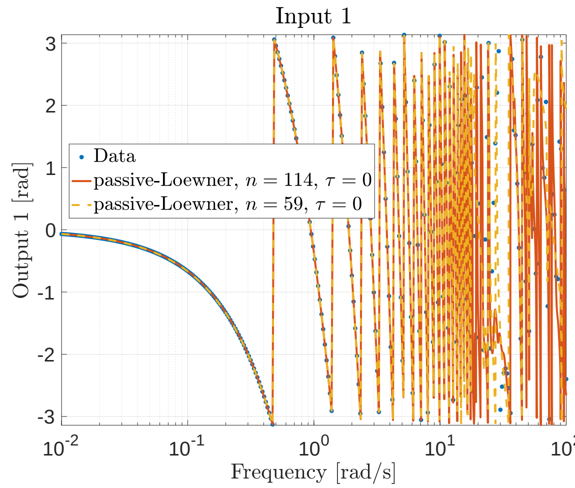

Figure 2. Frequency phase response of the original data ($N = 6,000$), the pH-Loewner models $n = 114$ and the reduced pH-Loewner ($n = 59$).

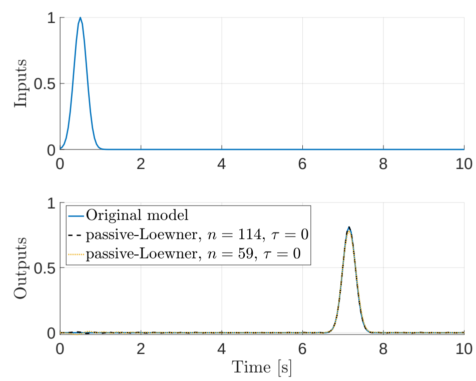

Figure 3. Reconstructed signals with low fidelity models with $n = 114$ or $n = 59$ (in each case, the pH model is delay free, i.e. $\tau = 0$).

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