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On Stokes-Lagrange and Stokes-Dirac representations for 1D distributed port-Hamiltonian systems ⋆

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Abstract: Port-Hamiltonian systems were recently extended to include implicitly defined energy and energy ports thanks to a (Stokes-)Lagrange subspace. Here, we study the equivalent port-Hamiltonian representations of two systems with damping, written using either a classical Hamiltonian or a Stokes-Lagrange subspace. Then, we study the Timoshenko beam and Euler-Bernoulli models, the latter being the flow-constrained version of the former, and show how they can be written using either a Stokes-Dirac or Stokes-Lagrange subspace related by a transformation operator. Finally, it is proven that these transformations commute with the flow-constraint projection operator.

Keywords: Distributed parameter systems, Implicit port-Hamiltonian systems, Constrained port-Hamiltonian systems, Dzektser equation, non-local viscous dissipation, Timoshenko beam, Euler-Bernoulli beam

1. INTRODUCTION

Port-Hamiltonian systems (pHs) have been developed for the modelling, (co-)simulation and control of complex multiphysics systems (see van der Schaft et al. (2014) for an introductive textbook). This framework has been extended to the case of distributed parameter systems (DPS) with boundary energy flows in the seminal paper van der Schaft and Maschke (2002). Since 2002, the literature on distributed pHs has grown considerably, with both theoretical and application papers (see Rashad et al. (2020), Skrepek (2021) and Philipp et al. (2023)).

PHs dynamics with algebraic constraints have been considered as well, leading to finite-dimensional pH Differential-Algebraic Equations systems (pH-DAEs, see Beattie et al. (2018)). These algebraic constraints arise either from the underlying Dirac structure between flow and effort variables, or from the constitutive equations, resulting in constraints between effort and energy state variables defined in some Lagrangian submanifold (see van der Schaft and Maschke (2020)).

Recently, examples of distributed parameter models given in implicit form have been considered in the pHs setting (see for instance Yaghi et al. (2022) for an implicit formulation of the Allen-Cahn equation, Jacob and Morris (2022) considering the Dzektser equation (seepage of underground water) or Heidari and Zwart (2019) considering a nanorod with non-local visco-elastic constitutive equations), along with structure-preserving numerical methods, see Bendimerad-Hohl et al. (2023). Maschke and van der Schaft (2023) extend boundary control pHs to a class of systems where the variational derivative of the Hamiltonian is replaced by a pair of reciprocal operators, generalizing – in the infinite-dimensional settings – the implicit definition of the energy by a so-called Stokes-Lagrange subspace associated to the reciprocal operators. Allowing the representation of some of the previously cited distributed parameter models given in implicit form. In particular, the example of an elastic rod with non-local elasticity relation.

In this paper, we propose first an application of this Stokes-Lagrange subspace approach to dissipative systems, in the case of local dissipation only (Dzekster equation, § 3.1), and in the case with local and non-local dissipative ports (see the nanorod example in § 3.2). In both cases, classical explicit formulations and representations using Stokes-Lagrange structures are proposed. Note that in the considered Stokes-Lagrange representations, spatial derivative operators are present inside the Hamiltonian as proposed for instance in Schöberl and Siuka (2014), where infinite-dimensional pHs are defined on jet bundles. In Preuster et al. (2024), Boussinesq, elastic rod and Allen-Cahn equations are considered as examples of such systems. The authors propose a lift in the jet space where the Hamiltonian density only depends on the extended state variable. Then, geometric formulations with Stokes-Dirac structures are applicable.

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In the second part of the paper, we derive similarly Stokes-Dirac and Stokes-Lagrange representations for the Timoshenko and Euler-Bernoulli beams. The Euler-Bernoulli model may be seen as flow-constrained version of the Stokes-Dirac Timoshenko beam and is therefore an example of constrained pHs with a definition of the energy via a Stokes-Lagrange subspace. Bijective operators which transform Stokes-Dirac formulations to Stokes-Lagrange ones are proposed for the Timoshenko beam model and for the Euler-Bernouilli reduced model. These operators are similar to transformations between DAEs and geometric representations analyzed in Mehrmann and van der Schaft (2023), in the finite-dimensional LTI case.

2. PRELIMINARY RESULTS

2.1 Variational derivatives

Theorem 1. Let $\mathcal{K} : D(\mathcal{K}) \subseteq L^2(\Omega,\mathbb{R}^n) \to L^2(\Omega,\mathbb{R}^m)$ be a closed and densely-defined linear operator, and let $\forall \alpha \in D(\mathcal{K}), \quad \mathcal{H}(\alpha) := \frac{1}{2}\int_\Omega |\mathcal{K}(\alpha)|^2 \,\mathrm{d}x$, be a functional.

Assume that the following abstract Green's identity holds for all $\alpha \in D(\mathcal{K})$, $\beta \in D(\mathcal{K}^\dagger)$

$$\int_\Omega \mathcal{K}(\alpha)\cdot\beta = \int_\Omega \alpha\cdot\mathcal{K}^\dagger(\beta) + \langle\gamma(\alpha),\mathcal{C}(\beta)\rangle_{\mathcal{U},\mathcal{Y}}\,, \quad (1)$$

where $\mathcal{K}^\dagger : D(\mathcal{K}^\dagger) \subseteq L^2(\Omega,\mathbb{R}^m) \to L^2(\Omega,\mathbb{R}^n)$ is another closed and densely-defined linear operator, called the *formal adjoint* of $\mathcal{K}$, $\gamma \in \mathcal{L}(D(\mathcal{K}),\mathcal{U})$ is a (boundary) control operator on the Hilbert space $\mathcal{U}$, and $\mathcal{C} \in \mathcal{L}(D(\mathcal{K}^\dagger),\mathcal{Y})$ its colocated observation operator, with $\mathcal{Y} := \mathcal{U}'$. Then, assuming that $C_c^\infty(\Omega,\mathbb{R}^m) \subset \ker(\gamma)$, the *weak* variational derivative of $\mathcal{H}$ with respect to α exists and is given as

$$\delta^w_\alpha \mathcal{H}(\alpha) = \mathcal{K}^\dagger(\mathcal{K}(\alpha)).$$

Proof. Let $\alpha \in D(\mathcal{K}), \varepsilon \in \mathbb{R}$, then, for all $\beta \in C_c^\infty(\Omega,\mathbb{R}^m) =: \mathcal{W}$, then, defining the weak variational derivative in the sense of distribution (thanks to (1))

$$\langle\delta^w_\alpha\mathcal{H}(\alpha),\beta\rangle_{\mathcal{W}',\mathcal{W}} := \int_\Omega \alpha\cdot\mathcal{K}^\dagger(\mathcal{K}(\beta))\,\mathrm{d}x,$$

gives the result.

Corollary 2. Given the Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$, defined on $H^1(\Omega,\mathbb{R}), H^2(\Omega,\mathbb{R})$, resp., by

$$\mathcal{H}_1(\alpha) = \frac{1}{2}\int_\Omega (\partial_x\alpha)^2\,\mathrm{d}x,\ \mathcal{H}_2(\alpha) = \frac{1}{2}\int_\Omega (\partial^2_{x^2}\alpha)^2\,\mathrm{d}x,$$

their weak variational derivatives are given, resp., by

$$\delta^w_\alpha H_1(\alpha) = -\partial^2_{x^2}\alpha, \quad \delta^w_\alpha H_2(\alpha) = \partial^4_{x^4}\alpha.$$

Proof. Take $\mathcal{K} = \partial_x$, $\mathcal{K}^\dagger = -\partial_x$ in the first case, and $\mathcal{K} = \mathcal{K}^\dagger = \partial^2_{x^2}$ in the second case.

2.2 Operator transposition

In the following, the superscript notations on a variable x^E, x^I refers to the Stokes-Dirac or Stokes-Lagrange representation of a system, respectively.

Let us consider $\mathcal{K}$ as before, two distributed second order tensors $\boldsymbol{\kappa} \in L^\infty(\Omega, M_n(\mathbb{R}))$, $\boldsymbol{\eta} \in L^\infty(\Omega, M_m(\mathbb{R}))$, with $\forall x \in \Omega$, $\boldsymbol{\kappa}(x) = \boldsymbol{\kappa}^\top(x) \geq \kappa > 0, \boldsymbol{\eta}(x) = \boldsymbol{\eta}^\top(x) \geq \eta > 0$ almost everywhere, and a pHs defined as

$$\begin{bmatrix}\partial_t\alpha^E_1\\ \partial_t\alpha^E_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & \mathcal{K}^\dagger\\ -\mathcal{K} & 0\end{bmatrix}}_{=:\mathcal{J}^E}\begin{bmatrix}e_1\\ e_2\end{bmatrix}, \quad \begin{cases}e^E_1 = \delta_{\alpha_1}\mathcal{H}^E,\\ e^E_2 = \delta_{\alpha_2}\mathcal{H}^E,\end{cases}$$

with the corresponding Hamiltonian defined as

$$\mathcal{H}^E = \frac{1}{2}\int_\Omega \alpha^E_1\cdot\boldsymbol{\kappa}\,\alpha^E_1 + \alpha^E_2\cdot\boldsymbol{\eta}\,\alpha^E_2\,\mathrm{d}x.$$

Now consider a second pHs defined as

$$\begin{bmatrix}\partial_t\alpha^I_1\\ \partial_t\alpha^I_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{bmatrix}}_{=:\mathcal{J}^I}\begin{bmatrix}e^I_1\\ e^I_2\end{bmatrix}, \quad \begin{cases}e^I_1 = \delta_{\alpha^I_1}\mathcal{H}^I,\\ e^I_2 = \delta^w_{\alpha^I_2}\mathcal{H}^I,\end{cases}$$

with the corresponding Hamiltonian defined as

$$\mathcal{H}^I = \frac{1}{2}\int_\Omega \alpha^I_1\cdot\boldsymbol{\kappa}\,\alpha^I_1 + \mathcal{K}(\alpha^I_2)\cdot\boldsymbol{\eta}\,\mathcal{K}(\alpha^I_2)\,\mathrm{d}x.$$

For the sake of readability, weak variational derivative will be denoted with the same symbol as usual variational derivative δ in the sequel of the paper. Finally, let us define an operator $\mathcal{G} : L^2(\Omega,\mathbb{R}^n)\times D(\mathcal{K}) \to L^2(\Omega,\mathbb{R}^n)\times L^2(\Omega,\mathbb{R}^m)$ and its †-companion

$$\mathcal{G} := \mathrm{Diag}\left[\mathrm{Id},\ \mathcal{K}\right], \quad \text{and} \quad \mathcal{G}^\dagger := \mathrm{Diag}\left[\mathrm{Id},\ \mathcal{K}^\dagger\right],$$

We then get the following theorem:

Theorem 3. The previously defined operator $\mathcal{G}$ allows passing from one representation to the other, the transformation being given as

$$\mathcal{G}\alpha^I = \alpha^E, \qquad e^I = \mathcal{G}^\dagger e^E, \qquad \mathcal{J}^E = \mathcal{G}\mathcal{J}^I\mathcal{G}^\dagger.$$

The two systems are said to be *equivalent*.

Proof. A direct computation gives the results.

Remark 4. Note that given such a transformation $\mathcal{G}$, each state $\alpha^I \in \ker(\mathcal{G})$ has zero energy, hence does not contribute to the Hamiltonian; hence, they are removed by $\mathcal{G}$. This explains why, when writing the wave equation in pH formulation, the deformation $\boldsymbol{\varepsilon}$ is used instead of the displacement w: indeed $\mathcal{H} = \frac{1}{2}\int_\Omega p^2 + |\mathbf{grad}(w)|^2\,\mathrm{d}x$, the Hamiltonian, only depends on the momentum p and deformation $\mathbf{grad}(w) = \boldsymbol{\varepsilon}$, hence the transformation removing **grad** from the Hamiltonian yields $(p,\boldsymbol{\varepsilon})$ as the state. And because the kernel of **grad** is the set of constant functions, rigid body motion is lost during the transformation.

3. TWO EXAMPLES WITH DAMPING

In § 3.1, the seepage model is considered, and in § 3.2, the example of the nanorod is studied. In this paper, the 1D domain Ω is defined as a bounded interval $\Omega = [a,b]$.

3.1 Dzektser

Following Dzektser (1972), let us consider the following seepage model of underground water in 1D

$$(\mathrm{Id} - \varepsilon^2\partial^2_{x^2})\,\partial_t h = \alpha\,\partial^2_{x^2}h - \beta\,\partial^4_{x^4}h,$$

with $\alpha > 0, \beta > 0$. The system admits a pH representation given as

$$\begin{bmatrix}(\mathrm{Id}-\varepsilon^2\partial^2_{x^2}) & 0 & 0\\ 0 & \mathrm{Id} & 0\\ 0 & 0 & \mathrm{Id}\end{bmatrix}\begin{bmatrix}h\\ F_\nabla\\ F_\Delta\end{bmatrix} = \underbrace{\begin{bmatrix}0 & \partial_x & -\partial^2_{x^2}\\ \partial_x & 0 & 0\\ \partial^2_{x^2} & 0 & 0\end{bmatrix}}_{:=\mathcal{J}}\begin{bmatrix}h\\ E_\nabla\\ E_\Delta\end{bmatrix}, \quad (2)$$

with the resistive relations $\begin{bmatrix} E_\nabla \\ E_\Delta \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}}_{:=\mathcal{R}} \begin{bmatrix} F_\nabla \\ F_\Delta \end{bmatrix}$. Following Maschke and van der Schaft (2023) one can define the Lagrange subspace operators $\mathcal{S} = (\mathrm{Id} - \varepsilon^2 \partial_{x^2}^2), \quad \mathcal{P} = \mathrm{Id}$, the Dirac structure operator $\mathcal{J}$ and the resistive structure operator $\mathcal{R}$. Finally, following (Maschke and van der Schaft, 2023, Section 5.), the Hamiltonian is given as

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\int_\Omega h\mathcal{S}^\dagger \mathcal{P} h \,\mathrm{d}x + \frac{1}{2}[\varepsilon^2 h \partial_x h]_a^b, \\ &= \frac{1}{2}\int_\Omega h^2 + \varepsilon^2 (\partial_x h)^2 \,\mathrm{d}x. \end{aligned}$$

Theorem 5. (Dzektser Power balance) The power balance related to (2) reads:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = [\alpha h \partial_x h - \beta h\, \partial_{x^3}^3 h + \beta \partial_x h\, \partial_{x^2}^2 h]_a^b \\ - \int_\Omega \left(a(\partial_x h)^2 + \beta(\partial_{x^2}^2 h)^2\right) \mathrm{d}x + [\varepsilon^2 \partial_x h\, \partial_t h]_a^b. \end{aligned}$$

Making use of tr as the Dirichlet trace operator, one can then define (f_∂, e_∂) the ***power*** *boundary port* related to the Stokes-Dirac structure as $f_\partial = \mathrm{tr}(\begin{bmatrix} h, & h, & \partial_x h \end{bmatrix})$, $e_\partial = \mathrm{tr}(\begin{bmatrix} \alpha\partial_x h, & -\beta\partial_{x^3}^3 h, & \beta\partial_{x^2}^2 h \end{bmatrix})$, and $(\chi_\partial, \varepsilon_\partial)$ the ***energy*** *boundary port* related to the Stokes-Lagrange subspace as $\chi_\partial = \mathrm{tr}(h), \quad \varepsilon_\partial = \mathrm{tr}(\varepsilon^2 \partial_x h)$. The power balance then reads:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} &= [f_\partial \cdot e_\partial + \frac{\mathrm{d}}{\mathrm{d}t}(\chi_\partial)\varepsilon_\partial]_a^b - \int_\Omega (\alpha F_\nabla^2 + \beta F_\Delta^2) \,\mathrm{d}x\,, \\ &\leq [f_\partial \cdot e_\partial + \frac{\mathrm{d}}{\mathrm{d}t}(\chi_\partial)\varepsilon_\partial]_a^b\,. \end{aligned}$$

Remark 6. The previous inequality is the extension to the case of lossy systems of the equality established in Maschke and van der Schaft (2023) for the case of lossless systems.

Remark 7. The energy boundary port $(\chi_\partial, \varepsilon_\partial)$ vanishes when $\varepsilon \to 0$, i.e., when the nonlocal term is removed, leading to a classical dissipative pHs.

3.2 Nanorod

Let us start by writing down both versions of the nanorod example and then comparing them.

Stokes-Dirac representation Following Heidari and Zwart (2019), the Hamiltonian of the system reads

$$\begin{aligned} \mathcal{H} := \frac{1}{2}\int_\Omega a^2 w^2 + \rho A \left(\partial_t w\right)^2 + \mu\rho A \left(\partial_{tx}^2 w\right)^2 \\ + \left(EA + \mu a^2\right)\left(\partial_x w\right)^2 \,\mathrm{d}x, \end{aligned}$$

and the state variable is given as

$$z := [w,\, \rho A \partial_t w,\, \mu\rho A \partial_{tx}^2 w,\, \partial_x w,\, N]^\top,$$

with w is the displacement, $\rho A \partial_t w$ the momentum density, $\mu\rho A\partial_{tx}^2$ the flow variable of the non locality, $\partial_x w$ the strain and N the stress resultant. Let us now define $\mathcal{E} := \mathrm{Diag}(\mathrm{Id}, \mathrm{Id}, \mathrm{Id}, \mathrm{Id}, 0)$ and

$$\mathcal{Q} := \mathrm{Diag}\left[a^2,\, \frac{1}{\rho A},\, \frac{1}{\mu\rho A},\, (EA + \mu a^2),\, \mathrm{Id}\right],$$

which allows us to rewrite the Hamiltonian $\mathcal{H}$ as $\mathcal{H} = \frac{1}{2}\int_\Omega z^\top \mathcal{E}^\top \mathcal{Q} z$, with the algebraic property $\mathcal{E}^\top\mathcal{Q} = \mathcal{Q}^\top\mathcal{E}$. Defining furthermore

$$\mathcal{J} := \begin{bmatrix} 0 & \mathrm{Id} & 0 & 0 & 0 \\ -\mathrm{Id} & 0 & 0 & 0 & \partial_x \\ 0 & 0 & 0 & -\mathrm{Id} & \mathrm{Id} \\ 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & \partial_x & -\mathrm{Id} & 0 & 0 \end{bmatrix},$$

$$\mathcal{R} := \mathrm{Diag}(0, b^2, \tau_d EA + \mu b^2, 0, 0),$$

where b is the damping coefficient of the viscoelastic layer and τ_d the viscous damping of the nanorod. The dynamics of the system is given by

$$\mathcal{E}\partial_t z = (\mathcal{J} - \mathcal{R})\, e, \qquad e = \mathcal{Q} z. \tag{3}$$

Stokes-Lagrange representation Let us now write this system with a Stokes-Lagrange subspace, i.e., as a state-space representation where differential operators are present in the Hamiltonian

$$\begin{bmatrix} \partial_t w \\ \partial_t \varepsilon \\ \partial_t p \\ f_d \\ f_\sigma \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \partial_x & 0 & 0 \\ -\mathrm{Id} & \partial_x & 0 & -\mathrm{Id} & \partial_x \\ 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \partial_x & 0 & 0 \end{bmatrix} \begin{bmatrix} F \\ \sigma \\ v \\ e_d \\ e_\sigma \end{bmatrix}, \tag{4}$$

with p the momentum, w the displacement, ε the strain, f_d and e_d the local and nonlocal dissipative ports, v the velocity, F the force of the media applied to the nanorod, σ the nonlocal stress, and e_d, e_σ the local and nonlocal efforts linked to the dissipative port. The constitutive relations are $F = aw,\ (\mathrm{Id} - \mu\,\partial_{x^2}^2)\sigma = E\varepsilon, v = \frac{p}{\rho A},\ e_d = b f_d,\ (\mathrm{Id} - \mu\,\partial_{x^2}^2) e_\sigma = \tau_d f_\sigma$. Notice that $(\mathrm{Id} - \mu\,\partial_{x^2}^2)$ is now inside the constitutive relations, and that two constitutive relations have become non-local.

Let us denote by $z = (w, \varepsilon, p)$ the state and $e_S = (F, \sigma, v)$ the effort variable corresponding to the storage port. We can now define the $\mathcal{S}$ and $\mathcal{P}$ matrices as

$$\mathcal{P} := \mathrm{Diag}\left[\mathrm{Id},\ (\mathrm{Id} - \mu\,\partial_{x^2}^2),\ \mathrm{Id}\right], \quad \mathcal{S} := \mathrm{Diag}\left[a,\ E,\ \frac{1}{\rho}\right]. \tag{5}$$

And we get the constitutive relation $\mathcal{S}^\dagger z = \mathcal{P}^\dagger e_S$. In particular, we have that $\mathcal{S}^\dagger\mathcal{P} = \mathcal{P}^\dagger\mathcal{S} \geq 0$, which shows that the corresponding Hamiltonian is non-negative. Moreover, the resistive matrices: $\mathcal{R}_L = \mathrm{Diag}\left[\mathrm{Id},\ (\mathrm{Id} - \mu\,\partial_{x^2}^2)\right]$, $\mathcal{R}_R = \mathrm{Diag}\left[b,\ \tau_d\right]$, yield an **implicit** resistive structure, $\mathcal{R}_L e_r = \mathcal{R}_R f_r$, with $\mathcal{R}_L^\dagger \mathcal{R}_R \geq 0$, where $f_r = (f_d, f_\sigma)^\top$ and $e_r = (e_d, e_\sigma)^\top$. Let us finally define the structure matrix

$$\mathcal{J} = \begin{bmatrix} 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \partial_x & 0 & 0 \\ -\mathrm{Id} & \partial_x & 0 & -\mathrm{Id} & \partial_x \\ 0 & 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \partial_x & 0 & 0 \end{bmatrix},$$

then, the system becomes

$$\begin{bmatrix} \partial_t z \\ f_r \end{bmatrix} = \mathcal{J} \begin{bmatrix} e_S \\ e_r \end{bmatrix}, \quad \begin{cases} \mathcal{R}_L e_r = \mathcal{R}_R f_r, \\ \mathcal{S}^\dagger z = \mathcal{P}^\dagger e_S. \end{cases}$$

Let us now write it using the *image representation* of the Lagrange subspace

$$\begin{bmatrix} \mathcal{P}\partial_t \xi \\ f_r \end{bmatrix} = \mathcal{J} \begin{bmatrix} e_S \\ e_r \end{bmatrix}, \quad \begin{cases} \mathcal{R}_L e_r = \mathcal{R}_R f_r, \\ e_S = \mathcal{S}\xi. \end{cases}$$

with $\mathcal{P}\xi = z$ being the latent space variable. Following (Maschke and van der Schaft, 2023, Section 5.), this representation allows us to define the Hamiltonian

$$\mathcal{H}^I := \frac{1}{2}\int_\Omega \xi^\top \mathcal{P}^\dagger \mathcal{S}\xi \,\mathrm{d}x + \frac{1}{2}[\mu E \xi_2 \partial_x \xi_2]_a^b,$$
$$= \frac{1}{2}\int_\Omega a{\xi_1}^2 + E{\xi_2}^2 + \mu E(\partial_x \xi_2)^2 + \frac{1}{\rho}{\xi_3}^2 \,\mathrm{d}x.$$

Theorem 8. (Nanorod) The power balance reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{\mathcal{I}} = [\,\mu E \partial_t \xi_2 \partial_x \xi_2 + \sigma v + e_\sigma v - \mu e_\sigma \partial_x e_\sigma\,]_a^b - \int_\Omega b f_d^2 + \frac{1}{\tau_d}(e_\sigma^2 + \mu(\partial_x e_\sigma)^2)\,\mathrm{d}x. \tag{6}$$

One can then identify the ***power*** *boundary ports* (f_∂, e_∂) and ***energy*** *boundary ports* $(\chi_\partial, \varepsilon_\partial)$ as

$$f_\partial = \mathrm{tr}(\begin{bmatrix} v, & v, & -\mu\frac{1}{\tau_d}\partial_x e_\sigma \end{bmatrix}),\ e_\partial = \mathrm{tr}([\sigma,\ e_\sigma,\ e_\sigma]),$$
$$\chi_\partial = \mathrm{tr}(\xi_2),\ \varepsilon_\partial = \mathrm{tr}(\mu E \partial_x \xi_2).$$

The power balance then reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^{\mathcal{I}} = [f_\partial \cdot e_\partial + \frac{\mathrm{d}}{\mathrm{d}t}(\chi_\partial)\varepsilon_\partial\,]_a^b - \int_\Omega b f_d^2 + \frac{1}{\tau_d}(e_\sigma^2 + \mu(\partial_x e_\sigma)^2)\mathrm{d}x,$$
$$\leq [\,f_\partial \cdot e_\partial + \frac{\mathrm{d}}{\mathrm{d}t}(\chi_\partial)\varepsilon_\partial\,]_a^b.$$

4. EQUIVALENT REPRESENTATIONS FOR CLASSICAL BEAM MODELS

In § 4.1, Stokes-Dirac representations both for the Timoshenko and the Euler-Bernoulli beam are recalled, then in § 4.2, Stokes-Lagrange representations are derived, thus recovering the examples treated in the jet bundle formalism first presented in Schöberl and Siuka (2014) for Timoshenko, and in (Schöberl and Schlacher, 2015, Example 5.1) for Euler-Bernoulli. Finally in § 4.3, the equivalence between these representations is proved, and the passage to the limit from Timoshenko to Euler-Bernoulli well understood in a common setting.

4.1 Stokes-Dirac representations

Timoshenko The Timoshenko beam equation reads (Ducceschi and Bilbao, 2019)

$$\begin{cases} \rho A \partial_t^2 w = T_0 \partial_{x^2}^2 w + A\kappa G \partial_x(\partial_x w - \phi), \\ \rho I \partial_t^2 \phi = EI \partial_{x^2}^2 \phi + A\kappa G(\partial_x w - \phi), \end{cases}$$

with ρ the mass density, A the cross section area, T_0 the tension, κ the sheer coefficient, G the shear modulus, I moment of inertia, E the young's modulus, w the transverse displacement and ϕ the shear angle. Let us firstly write it as an explicit pHs

$$\begin{bmatrix} \partial_t \varepsilon_w \\ \partial_t p_w \\ \partial_t \varepsilon_\phi \\ \partial_t p_\phi \\ \partial_t \varepsilon_{w,\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \partial_x & 0 & 0 & 0 \\ \partial_x & 0 & 0 & 0 & \partial_x \\ 0 & 0 & 0 & \partial_x & 0 \\ 0 & 0 & \partial_x & 0 & \mathrm{Id} \\ 0 & \partial_x & 0 & -\mathrm{Id} & 0 \end{bmatrix}}_{=:\mathcal{J}^E} \begin{bmatrix} \sigma_w \\ v \\ \sigma_\phi \\ \omega \\ N \end{bmatrix}, \tag{7}$$

with p_w the linear momentum, p_ϕ the angular momentum, $\varepsilon_w = \partial_x w$ the deformation, $\varepsilon_\phi = \partial_x \phi$ the spatial derivative of the shear angle, and $\varepsilon_{w,\phi} = \partial_x w - \phi$ the difference between the deformation and the shear angle. With the following constitutive relations: $v = \frac{p_w}{\rho A}$, $\sigma_w = T_0 \varepsilon_w$, $\omega = \frac{p_\phi}{\rho I}$ $\sigma_\phi = EI\varepsilon_\phi$, $N = A\kappa G \varepsilon_{w,\phi}$. And Hamiltonian

$$\mathcal{H}^E := \frac{1}{2}\int_\Omega \Big(\frac{{p_w}^2}{\rho A} + T_0 \varepsilon_w^2 + \frac{{p_\phi}^2}{\rho I} + EI\varepsilon_\phi^2 + A\kappa G \varepsilon_{w,\phi}^2\Big)\,\mathrm{d}x.$$

Moreover, one can define the matrices

$$\mathcal{S}^E = \mathrm{Id}, \quad \mathcal{P}^E = \mathrm{Diag}\left(T_0,\ \frac{1}{\rho A},\ EI,\ \frac{1}{\rho I},\ A\kappa G\right),$$

with $\mathcal{P}^{E\top}\mathcal{S}^E = \mathcal{S}^{E\top}\mathcal{P}^E > 0$. Finally one gets the following power balance:

Theorem 9. (Timoshenko power balance - Stokes-Dirac case) The power balance of (7) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^E = [v\sigma_w + vN + \omega\sigma_\phi]_a^b. \tag{8}$$

One can identify the ***power*** *boundary port* variables $f_\partial = \mathrm{tr}([v\ v\ \omega]^\top), e_\partial = \mathrm{tr}([\sigma_w\ N\ \sigma_\phi]^\top)$, tr being the trace operator; yielding $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^E = [f_\partial \cdot e_\partial]_a^b$.

Euler–Bernoulli To reduce the Timoshenko model to the Euler–Bernoulli model, one simply needs to add two constraints by setting $\partial_t p_\phi$ and $\partial_t \varepsilon_{w,\phi}$ to zero in (7)

$$\begin{bmatrix} \partial_t \varepsilon_w \\ \partial_t p_w \\ \partial_t \varepsilon_\phi \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \mathcal{J}^E \begin{bmatrix} \sigma_w \\ v \\ \sigma_\phi \\ \omega \\ N \end{bmatrix}. \tag{9}$$

One can then define the constrained Hamiltonian

$$\mathcal{H}_c^E = \frac{1}{2}\int_\Omega \frac{{p_w}^2}{\rho A} + T_0\varepsilon_w^2 + EI\varepsilon_\phi^2\,\mathrm{d}x, \tag{10}$$

and get the following power balance

Theorem 10. (Euler-Bernoulli power balance - Stokes-Dirac case) The power balance of (9) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_c^E = [v\sigma_w - v\partial_x\sigma_\phi + \sigma_\phi\partial_x v]_a^b. \tag{11}$$

One can identify the ***power*** *boundary port* variables $f_\partial^c = \mathrm{tr}([v\ v\ \partial_x v]^\top), e_\partial^c = \mathrm{tr}([\sigma_w\ -\partial_x\sigma_\phi\ \sigma_\phi]^\top)$. This yields $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^E = [f_\partial^c \cdot e_\partial^c]_a^b$.

Remark 11. One can solve the constraints analytically: $\partial_x\sigma_\phi = -N$, $\partial_x v = \omega$. Which yields the following reduced system

$$\begin{bmatrix} \partial_t \varepsilon_w \\ \partial_t \varepsilon_\phi \\ \partial_t p_w \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_{x^2}^2 \\ \partial_x & -\partial_{x^2}^2 & 0 \end{bmatrix}}_{=:\mathcal{J}_r^E} \begin{bmatrix} \sigma_w \\ \sigma_\phi \\ v \end{bmatrix}. \tag{12}$$

Note that by hypothesis, $\partial_t \varepsilon_{w,\phi} = \partial_t(\partial_x\phi - w) = 0$, hence $\partial_x\phi - w$ is constant over time, however, this constraint is not written explicitly in the previous set of equations.

4.2 Stokes-Lagrange representations

Timoshenko Let us now write the Timoshenko beam equation in Stokes-Lagrange form, i.e. by putting the differential operators in the Hamiltonian.

$$\begin{bmatrix} \partial_t w \\ \partial_t p_w \\ \partial_t \phi \\ \partial_t p_\phi \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Id} & 0 & 0 \\ -\mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} \\ 0 & 0 & -\mathrm{Id} & 0 \end{bmatrix} \begin{bmatrix} \delta_w \mathcal{H}^I \\ \delta_{p_w} \mathcal{H}^I \\ \delta_\phi \mathcal{H}^I \\ \delta_{p_\phi} \mathcal{H}^I \end{bmatrix}, \tag{13}$$

with Hamiltonian

$$\mathcal{H}^I := \frac{1}{2}\int_\Omega \Big(\frac{1}{\rho A}p_w^2 + \frac{1}{\rho I}p_\phi^2 + T_0(\partial_x w)^2 + EI(\partial_x\phi)^2 + A\kappa G(\partial_x w - \phi)^2\Big)\,\mathrm{d}x. \tag{14}$$

Note that boundary terms will then appear in the constitutive relations. Moreover, this formulation allows for direct access to the displacement w instead of its gradient ε_w. Let us use Corollary 2 to write the constitutive relations

$$\begin{cases} \delta_w\mathcal{H}^I = -\partial_x(T_0\partial_x w) - \partial_x(A\kappa G(\partial_x w - \phi)), & \delta_{p_w}\mathcal{H}^I = \dfrac{p_w}{\rho A}, \\ \delta_\phi\mathcal{H}^I = -\partial_x(EI\partial_x\phi) - A\kappa G(\partial_x w - \phi), & \delta_{p_\phi}\mathcal{H}^I = \dfrac{p_\phi}{\rho I}. \end{cases}$$

We can now define the structure matrix $\mathcal{J}$ and the Lagrange subspace operators $\mathcal{S}$ and $\mathcal{P}$ as

$$\mathcal{J}^I = \begin{bmatrix} 0 & \mathrm{Id} & 0 & 0 \\ -\mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} \\ 0 & 0 & -\mathrm{Id} & 0 \end{bmatrix}, \quad \mathcal{P}^I = \begin{bmatrix} \mathrm{Id} & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & \mathrm{Id} \end{bmatrix},$$

$$\mathcal{S}^I = \begin{bmatrix} \mathcal{S}_{1,1} & 0 & \partial_x(A\kappa G\cdot) & 0 \\ 0 & \dfrac{1}{\rho A} & 0 & 0 \\ -A\kappa G\partial_x\cdot & 0 & \mathcal{S}_{3,3} & 0 \\ 0 & 0 & 0 & \dfrac{1}{\rho I} \end{bmatrix},$$

with $\mathcal{S}^I_{1,1} := -\partial_x(T_0\partial_x\cdot) - \partial_x(A\kappa G(\partial_x\cdot))$ and $\mathcal{S}^I_{3,3} := -\partial_x(EI\partial_x\cdot) + A\kappa G\,\mathrm{Id}$.

Then, writing $z^I = (w, p_w, \phi, p_\phi)^\top$, and $e^I = (\delta_w\mathcal{H}, \delta_{p_w}\mathcal{H}, \delta_\phi\mathcal{H}, \delta_{p_\phi}\mathcal{H})^\top$, we get

$$\begin{cases} \partial_t z^I = \mathcal{J}^I e^I, \\ \mathcal{P}^{I\dagger} e^I = \mathcal{S}^{I\dagger} z^I, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{P}^I\partial_t\xi^I = \mathcal{J}^I e^I, \\ e^I = \mathcal{S}^I\xi^I, \end{cases}$$

the latter being the image representation. Note that, $\mathcal{S}^\dagger\mathcal{P} = \mathcal{P}^\dagger\mathcal{S} \geq 0$, which comes from the non-negativity of the Hamiltonian (14). Moreover, defining $\mathcal{R} := [\mathcal{P}\ \mathcal{S}]^\top$, since $\mathcal{S}$ is invertible, the associated polynomial matrix of the operator $R(s)$ satisfies $\mathrm{rank}(R(s)) = 4\,,\ \forall s \in \mathbb{C}$, giving the maximal reciprocity of the operator $\mathcal{R}$ (see Maschke and van der Schaft (2023) for details).

Theorem 12. (Timoshenko power balance - Stokes-Lagrange case) The power balance of (14) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^I = [\,\partial_t w\,(T_0\,\partial_x w + A\kappa G\,(\partial_x w - \phi)) + \partial_t\phi\,EI\partial_x\phi\,]_a^b.$$

One can identify the ***energy*** *boundary port* variable

$$\chi_\partial = \mathrm{tr}([w,\ \ w,\ \ \phi]^\top),$$
$$\varepsilon_\partial = \mathrm{tr}([T_0\,\partial_x w,\ \ A\kappa G\,(\partial_x w - \phi),\ \ EI\,\partial_x\phi]^\top),$$

this yields $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^I = [\,\frac{\mathrm{d}}{\mathrm{d}t}(\chi_\partial)\cdot\varepsilon_\partial\,]_a^b$.

Euler–Bernoulli Let us reduce the previously defined system by putting $\frac{1}{\rho A}p_\phi^2 = 0$ and $A\kappa G(\partial_x w - \phi)^2 = 0$ in the Hamiltonian (14). One gets: $\partial_t p_\phi = 0, \quad \phi = \partial_x w$. This allows us to define the constrained Hamiltonian as

$$\mathcal{H}_c^I = \frac{1}{2}\int_\Omega \left(\frac{1}{\rho A}p_w^2 + T_0(\partial_x w)^2 + EI(\partial^2_{x^2} w)^2\right)\mathrm{d}x.$$

The constrained dynamic reads

$$\begin{bmatrix} \mathrm{Id} & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ \partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \partial_t w \\ \partial_t p_w \\ \partial_t\phi \\ \partial_t p_\phi \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Id} & 0 & 0 \\ -\mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id} \\ 0 & 0 & -\mathrm{Id} & 0 \end{bmatrix}\begin{bmatrix} \delta_w\mathcal{H}^I \\ \delta_{p_w}\mathcal{H}^I \\ \delta_\phi\mathcal{H}^I \\ \delta_{p_\phi}\mathcal{H}^I \end{bmatrix}. \tag{15}$$

Theorem 13. (Euler–Bernoulli power balance - Stokes-Lagrange case)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_c^I = [\,\partial_t w\,(T_0\partial_x w - \partial_x(EI\partial^2_{x^2}w)) + \partial_t(\partial_x w)\,EI\,\partial^2_{x^2}w\,]_a^b.$$

One can identify the ***energy*** *boundary port* variable

$$\chi_\partial^c = \mathrm{tr}([w,\ \ w,\ \ \partial_x w]^\top),$$
$$\varepsilon_\partial^c = \mathrm{tr}([T_0\,\partial_x w,\ \ -\partial_x(EI\,\partial^2_{x^2}w),\ \ EI\,\partial^2_{x^2}w]^\top),$$

this yields $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^I = [\,\frac{\mathrm{d}}{\mathrm{d}t}(\chi_\partial^c)\cdot\varepsilon_\partial^c\,]_a^b$.

Remark 14. Solving the two constraints $\partial_t p_\phi$ and $\partial_x w = \phi$ yields the reduced unconstrained system

$$\begin{bmatrix} \partial_t w \\ \partial_t p_w \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=:\mathcal{J}_r^I}\begin{bmatrix} \delta_w\mathcal{H}_c^I \\ \delta_{p_w}\mathcal{H}_c^I \end{bmatrix}, \tag{16}$$

together with the constitutive relations:

$$\begin{cases} \delta_w\mathcal{H}_c^I = -\partial_x(T_0\partial_x w) + \partial^2_{x^2}(EI\partial^2_{x^2}w), \\ \delta_{p_w}\mathcal{H}_c^I = \dfrac{p_w}{\rho A}. \end{cases}$$

4.3 From Stokes-Lagrange to Stokes-Dirac formulations

Timoshenko In this subsection, we will denote by z^E, e^E the state and effort variables in the Stokes-Dirac formulation and by z^I, e^I the state and effort in the Stokes-Lagrange formulation. $\mathcal{J}^E, \mathcal{S}^E, \mathcal{P}^E$ denote the structure and constitutive matrices in the Stokes-Dirac case and $\mathcal{J}^I, \mathcal{S}^I, \mathcal{P}^I$ the structure and constitutive matrices in the Stokes-Lagrange case. Let us now describe a procedure that allows to pass from the Stokes-Lagrange to the Stokes-Dirac representation, this procedure is similar to the one described in (Mehrmann and van der Schaft, 2023, Section 6.) but now makes use of differential operators instead of matrices. First let us define a transformation operator $\mathcal{G}$ from $(H^1 \cap L^2_0) \times L^2 \times H^1 \times L^2$ to $(L^2)^5$, where L^2_0 is the space of zero mean functions, $\int_\Omega w = 0$, as

$$\mathcal{G} := \begin{bmatrix} \partial_x & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ 0 & 0 & \partial_x & 0 \\ 0 & 0 & 0 & \mathrm{Id} \\ \partial_x & 0 & -\mathrm{Id} & 0 \end{bmatrix}.$$

Its inverse is actually well-defined, thanks to Poincaré-Wirtinger's inequality, since $\int_\Omega w = 0$. Physically speaking, it means that one neglects rigid body motions of the beam. The inverse $\mathcal{F}$ of $\mathcal{G}$ is given by

$$\mathcal{F}(z) := z \mapsto \begin{bmatrix} x \mapsto \left(\displaystyle\int_a^x z_2\,\mathrm{d}y - \frac{1}{b-a}\int_a^b z_2\,\mathrm{d}y\right) \\ z_1 \\ z_2 - z_5 \\ z_3 \end{bmatrix}. \tag{17}$$

A direct computation then yields $\mathcal{G}z^I = z^E, \quad e^I = \mathcal{G}^\dagger e^E, \quad z^I = \mathcal{F}z^E, \quad \mathcal{F}^\dagger e^I = e^E$.

Note that a *geometric* interpretation might be useful to understand these transformations, since states are often

described as vectors and co-states as covectors, hence they are contravariant and covariant, respectively. Then, one can compute the time derivative in the Stokes-Lagrange case and compare it to the Stokes-Dirac case

$$\partial_t \mathcal{G}(z^I) = \mathcal{G}\mathcal{J}^I e^I = \mathcal{G}\mathcal{J}^I\mathcal{G}^\dagger e^E.$$

Hence $\mathcal{G}\mathcal{J}^I\mathcal{G}^\dagger = \mathcal{J}^E$. And, regarding the Lagrange subspace $\mathcal{S}^{E\top} z^E = \mathcal{P}^{E\top} e^E$. Let us premultiply this equation by $\mathcal{G}^\dagger$ and replace the state and effort variables by their Stokes-Dirac counterparts. We get to

$$\mathcal{G}^\dagger\mathcal{S}^{E\top}\mathcal{G} z^I = \mathcal{G}^\dagger\mathcal{P}^{E\top}\mathcal{F}^\dagger e^I,$$

and we have $\mathcal{G}^\dagger\mathcal{S}^{E\top}\mathcal{G} = \mathcal{S}^{I\dagger}$ and $\mathcal{G}^\dagger\mathcal{P}^{E\top}\mathcal{F}^\dagger = \mathcal{P}^{I\dagger}$.

From Timoshenko to Euler–Bernoulli Let us denote by $\Pi^E := \mathrm{Diag}\,[\mathrm{Id},\ \mathrm{Id},\ \mathrm{Id},\ 0,\ 0]$, the (singular) operator that allows to constrain system (7); when pre-multiplying the left-hand side of equation (7) with Π^E, one gets the constrained system (9). In order to get the Stokes-Lagrange counterpart Π^I, let us apply $\mathcal{G}$ to pass from (13) to (7), then Π^E to pass from (7) to (9), and finally $\mathcal{F}$ to pass from (9) to (15). Therefore, let us compute $\mathcal{F}\,\Pi^E\,\mathcal{G}$

$$\Pi^I := \mathcal{F}\,\Pi^E\,\mathcal{G} = \begin{bmatrix} \mathrm{Id} & 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 & 0 \\ \partial_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, one can represent these transformations as a commutative diagram, shown in Fig. 1.

$$\begin{array}{ccc} (7),\ \mathcal{H}^E & \xleftrightarrow[\mathcal{G},\mathcal{F}]{} & (13),\ \mathcal{H}^I \\ \Pi^E\downarrow & & \downarrow\Pi^I \\ (9),\ \mathcal{H}^E_c & \xleftrightarrow[\mathcal{G},\mathcal{F}]{} & (15),\ \mathcal{H}^I_c \end{array}$$

Fig. 1. Transformations between beam models and representations

Euler–Bernoulli Let us now apply the same procedure to the Euler–Bernoulli *reduced* case, i.e., let us pass from (12) to (16). Let us denote by $\mathcal{G}_r := \begin{bmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \partial_x & \partial^2_{x^2} \end{bmatrix}^\top$, the transformation operator between the two reduced systems. Then, assuming $\varepsilon_\phi = \partial_x w$, i.e., on the subspace where this constraint is satisfied, we have

$$\mathcal{G}_r \begin{bmatrix} w \\ p_w \end{bmatrix} = \begin{bmatrix} \varepsilon_w \\ \varepsilon_\phi \\ p_w \end{bmatrix}, \quad \text{and} \quad \mathcal{G}_r^\dagger \begin{bmatrix} \sigma_w \\ \sigma_\phi \\ v \end{bmatrix} = \begin{bmatrix} \delta_w \mathcal{H}^I \\ \delta_{p_w} \mathcal{H}^I \end{bmatrix}.$$

Additionally, one has: $\mathcal{G}_r\mathcal{J}^I_r\mathcal{G}^\dagger_r = \mathcal{J}^E_r$.

5. CONCLUSION & OUTLOOK

Stokes-Lagrange representations have been proposed for two examples of pHs, respectively with local and non-local dissipation. The corresponding Hamiltonian and power boundary and energy boundary port-variables have been derived. Stokes-Lagrange representations have also been derived for the Timoshenko and the Euler-Bernoulli models. Two bijective transformations between the corresponding Stokes-Dirac and Stokes-Lagrange models have been exhibited. Since the Euler-Bernoulli beam is a flow-constrained version of the Timoshenko beam, it has been checked that the corresponding projection operators commute with the transformations between Stokes-Dirac and Stokes-Lagrange representations of both models.

We will now consider extending these results to 2D (Dzekster seapage model, Reissner-Mindlin and Kirchoff-Love plate models) and 3D (Maxwell equations) examples. Another avenue we want to explore concerns the comparative analysis of Stokes-Dirac and Stokes-Lagrange formulations, in terms of their numerical properties.

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