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Explicit structure-preserving discretization of port-Hamiltonian systems with mixed boundary control [*]

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Abstract:
In this contribution, port-Hamiltonian systems with non-homogeneous mixed boundary conditions are discretized in a structure-preserving fashion by means of the Partitioned FEM. This means that the power balance and the port-Hamiltonian structure of the continuous equations is preserved at the discrete level. The general construction relies on a weak imposition of the boundary conditions by means of the Hellinger-Reissner variational principle, as recently proposed in [Thoma et al., 2021]. The case of linear hyperbolic wave-like systems, including the elastodynamic problem and the Maxwell equations in 3D, is then illustrated in detail. A numerical example is worked out on the case of the wave equation.

Keywords: Port-Hamiltonian systems (pHs), Partitioned Finite Element Method (PFEM), Mixed Boundary Control.

1. INTRODUCTION

Port-Hamiltonian systems (pHs) have been extended to distributed parameter systems in van der Schaft and Maschke (2002), and since then are an active field of research, see Rashad et al. (2020) for an overview. One important topic is the structure-preserving discretization of such dynamical systems. One of the interesting and promising methods is the Partitioned Finite Element Method (PFEM), see Cardoso-Ribeiro et al. (2021), which can be seen as an extension of the classical Mixed Finite Element Method to systems with boundary control and observation. In the case of uniform boundary control, one of the variables to integrate by part was chosen accordingly, and the resulting finite-dimensional system was a pH-ODE, i.e. an explicit system. In the case of non-uniform or mixed boundary control, whatever the choice of partition, the obtained system was a pH-DAE, i.e. an implicit system, see Brugnoli et al. (2020). Though a rich theory for such systems is available, see e.g. van der Schaft (2013); Beattie et al. (2018) and references therein, from the numerical point of view it can prove more appealing to deal with explicit ODEs than with implicit DAEs (see e.g. Serhani et al. (2019); Brugnoli et al. (2021) for the example of the heat equation, or Haine and Matignon (2021) for the case of the nonlinear incompressible Navier-Stokes equation in 2D). The explicit formulation removes the need to impose the boundary conditions strongly. This is of particular interest for some finite element families, for which it is highly non trivial, for example the Argyris or Bell H^2 conforming finite elements. Recently, in Thoma and Kotyczka (2021), a new version of PFEM has been introduced for mixed boundary control on the example of linear elastodynamics, which gives rise to a pH-ODE: the method is based on the so-called Hellinger-Reissner principle, see Arnold (1990) for an original presentation of the idea, and Lu et al. (2019) for a more recent comparison of different methods. Moreover, an accurate presentation of the strong or weak imposition of Dirichlet boundary conditions can be found in Benner and Heiland (2015).

The goal of this contribution is to provide a detailed general formulation of the Hellinger-Reissner principle on linear distributed pHs first, in § 2. Then, a variety of practical examples stemming from engineering applications are considered in § 3: the wave equation, elastodynamics, and Maxwell's equations in 3D. Finally, § 4 is devoted to numerical considerations: the test case on the wave equation, together with a discussion on the pros and cons of the approach from the numerical point of view.

2. WEAK IMPOSITION OF MIXED BOUNDARY CONDITIONS: A GENERAL RESULT

2.1 Preliminaries

Suppose that $\Omega \subset \mathbb{R}^d$ with $d = \{1, 2, 3\}$ is a bounded connected set and that its boundary $\partial\Omega$ is divided into a partition of two subsets that satisfy $\partial\Omega = \overline{\Sigma}_1 \cup \overline{\Sigma}_2$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$. Each Σ_i is associated to a specific kind of boundary conditions, given by a boundary trace operator γ_i.

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Let us introduce an abstract functional framework. Given a differential operator

$$L : L^2(\Omega, \mathbb{A}) \to L^2(\Omega, \mathbb{B}),$$

where $L^2(\Omega, \mathbb{A})$, resp. $L^2(\Omega, \mathbb{B})$, is the space of square integrable functions from Ω to the space $\mathbb{A}$, resp. $\mathbb{B}$. The space $\mathbb{A}$ indicates either a scalar, a vector, a tensor field or a Cartesian product of those and analogously for $\mathbb{B}$. In particular in the following we will use the following notation for the space of d-dimensional vectors and $d \times d$ symmetric matrices

$$\mathbb{V} := \mathbb{R}^d, \qquad \mathbb{S} := \mathbb{R}^{d\times d}_{\mathrm{sym}}.$$

The operator L, being unbounded, has domain

$$D(L) = \{u \in L^2(\Omega, \mathbb{A}) |\ Lu \in L^2(\Omega, \mathbb{B})\}.$$

Furthermore, we denote by $L^\dagger$, defined on

$$D(L^\dagger) = \{u \in L^2(\Omega, \mathbb{B}) |\ L^\dagger u \in L^2(\Omega, \mathbb{A})\},$$

a formal adjoint operator of L with respect to the γ_i operators, for $i \in \{1, 2\}$, and the partition of $\partial\Omega$, *i.e.* an operator satisfying

$$\left\langle Le_1,\, e_2 \right\rangle_{L^2(\Omega,\mathbb{B})} = \left\langle e_1,\, L^\dagger e_2 \right\rangle_{L^2(\Omega,\mathbb{A})}, \forall e_i \in \operatorname{Ker}\left(\gamma_i^{\Sigma_i}\right),$$

where $\gamma_i^{\Sigma_i}$ is the restriction of the operator γ_i to Σ_i.

The first assumption introduces the abstract integration by parts formula of fundamental importance.

Assumption 1. The operators L and $L^\dagger$ are assumed to satisfy the following integration by parts formula

$$\left\langle Le_1,\, e_2 \right\rangle_{L^2(\Omega,\mathbb{B})} - \left\langle e_1,\, L^\dagger e_2 \right\rangle_{L^2(\Omega,\mathbb{A})} = \left\langle \gamma_1 e_1 \,|\gamma_2 e_2 \right\rangle_{V_\partial, V_\partial'}, \tag{1}$$

where $\langle \cdot\, |\cdot \rangle_{V_\partial, V_\partial'}$ denotes the duality product between the boundary space V_∂ and its dual V_∂'.

This assumption essentially says that L^*, the adjoint of $L|_{\gamma_1^{\Sigma_1}}$, can be continuously extended to $D(L^\dagger)$ by the sum of a differential operator, namely $L^\dagger$, and a boundary term (this relies on the decomposition $L = A + BG$ in boundary control systems theory (Tucsnak and Weiss, 2009, Chapter 10)). In practice, it is derived from the celebrated Stokes divergence theorem.

The integration by parts formula (1) is valid $\forall e_1 \in D(L)$, $\forall e_2 \in D(L^\dagger)$ with the domain of $L^\dagger$ given by

$$D(L^\dagger) = \{u \in L^2(\Omega, \mathbb{B}) |\ L^\dagger u \in L^2(\Omega, \mathbb{A})\}.$$

This integration by parts formula is verified for all the subsequent examples.

As an example, one may consider the gradient operator

$$\operatorname{grad} : L^2(\Omega) \to L^2(\Omega, \mathbb{V}),$$

with domain

$$D(\operatorname{grad}) = H^1(\Omega) := \{u \in L^2(\Omega) |\ \operatorname{grad} u \in L^2(\Omega, \mathbb{V})\}. \tag{2}$$

Operators γ_i are then the well-known Dirichlet trace

$$\gamma_0 u = u|_{\partial\Omega} \tag{3}$$

and normal trace operator on vector fields $\mathbf{u}$

$$\gamma_n \mathbf{u} = \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega}. \tag{4}$$

The formal adjoint is $L^\dagger = -\operatorname{div}$ with domain

$$D(\operatorname{div}) = H^{\operatorname{div}}(\Omega) := \{\mathbf{u} \in L^2(\Omega, \mathbb{V}) |\ \operatorname{div} \mathbf{u} \in L^2(\Omega)\}, \tag{5}$$

and (1) is nothing but Green's formula

$$\int_\Omega \operatorname{grad} u \cdot \mathbf{v} = -\int_\Omega u \operatorname{div} \mathbf{v} + \left\langle \gamma_0 u \,|\gamma_n \mathbf{v} \right\rangle_{H^{\frac12}(\partial\Omega), H^{-\frac12}(\partial\Omega)}. \tag{6}$$

2.2 Boundary control operator and functional spaces

Since mixed boundary control systems are considered, an important operator has already appeared in the definition of the formal adjoint, namely the boundary control operator

$$G_u = \begin{bmatrix} \gamma_1^{\Sigma_1} & 0 \\ 0 & \gamma_2^{\Sigma_2} \end{bmatrix} \in \mathcal{L}(D(L) \times D(L^\dagger), \mathcal{U}_1 \times \mathcal{U}_2) \tag{7}$$

where $\mathcal{U}_i,\ i \in \{1, 2\}$ are the control spaces and $\mathcal{L}(X, Y)$ denotes the set of bounded linear operators from X to Y.

A natural but difficult question concerns the close relation between V_∂, V_∂' and the boundary control spaces $\mathcal{U}_i$. Since G_u is diagonal, it is tempting to consider a splitting by a simple cartesian product such as $\mathcal{U}_1 \times \mathcal{U}_2 = H^{\frac12}(\Sigma_1) \times H^{-\frac12}(\Sigma_2)$. However, care must be taken, keeping our goal in mind.

On the one hand, if the objective is to prove the well-posedness of solution, then compatibility relations at the interface(s) $\overline{\Sigma_1} \cap \overline{\Sigma_2}$ must be fulfilled by the controls u_1 and u_2, and the restriction to the range of $\gamma_i^{\Sigma_i}$ is not sufficient in general; however, we do not go further in this direction. The interested reader may consult Nguyen and Raymond (2015) for a complete characterization of these spaces in the case of the Navier-Stokes equations.

On the other hand, it can be sufficient to consider the above splitting for finite element conformity considerations, as in the present work, since obviously the range of the restriction $\gamma_i^{\Sigma_i}$ contains the suitable spaces $\mathcal{U}_i$ for (possible) well-posedness. In other words, if the boundary control system is well-posed for control spaces $\mathcal{U}_i$, it is true that

$$\mathcal{U}_i \subset \operatorname{Range}(\gamma_i^{\Sigma_i}).$$

This implies the following splitting of boundary duality product

$$\begin{aligned} \left\langle \gamma_1 e_1 \,|\gamma_2 e_2 \right\rangle_{V_\partial, V_\partial'} =& \langle \gamma_1^{\Sigma_1} e_1 \,|\gamma_2^{\Sigma_1} e_2 \rangle_{V_{\partial,1}, V_{\partial,1}'} \\ &+ \langle \gamma_1^{\Sigma_2} e_1 \,|\gamma_2^{\Sigma_2} e_2 \rangle_{V_{\partial,2}, V_{\partial,2}'}, \\ =& \langle u_1 \,|y_1 \rangle_{V_{\partial,1}, V_{\partial,1}'} + \langle y_2 \,|u_2 \rangle_{V_{\partial,2}, V_{\partial,2}'}, \\ =& \langle u_1 \,|y_1 \rangle_{\mathcal{U}_1, \mathcal{Y}_1} + \langle y_2 \,|u_2 \rangle_{\mathcal{Y}_2, \mathcal{U}_2}. \end{aligned} \tag{8}$$

The input and output functional spaces are defined according with the splitting of the boundary duality pairing

$$\begin{aligned} \mathcal{U}_1 \subset V_{\partial,1} &:= \operatorname{Range}(\gamma_1^{\Sigma_1}), \qquad \mathcal{Y}_1 \supset V_{\partial,1}', \\ \mathcal{U}_2 \subset V_{\partial,2}' &:= \operatorname{Range}(\gamma_2^{\Sigma_2}), \qquad \mathcal{Y}_2 \supset V_{\partial,2}. \end{aligned}$$

Care must be taken that there is no inclusion relation between $V_\partial := V_{\partial,1} \times V_{\partial,2}$ and $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, nor between V_∂ and $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$.

At the discrete level, this means that finite element conformity relies on $V_{\partial,1}$ and $V_{\partial,2}$ separately (the most regular boundary spaces), while the compatibility condition needed for well-posedness is postponed to suitable choices of a couple of controls $(u_1, u_2) \in V_{\partial,1} \times V_{\partial,2}'$ in practice.

Remark 1. From now on, we take for granted that compatibility conditions are met, and will not discriminate between $\langle \cdot\,|\cdot\rangle_{\mathcal{U}_1,\mathcal{Y}_1}$ and $\langle \cdot\,|\cdot\rangle_{V_{\partial,1},V'_{\partial,1}}$, and similarly for $\langle \cdot\,|\cdot\rangle_{\mathcal{Y}_2,\mathcal{U}_2}$ and $\langle \cdot\,|\cdot\rangle_{V_{\partial,2},V'_{\partial,2}}$. Consequently, in what follows, L^2 inner-product over the domain will be denoted by $\langle \cdot,\,\cdot\rangle_\Omega$, i.e. the specification of the nature of the variables is dropped for notational simplicity. And the notation for the boundary duality products will be simplified as $\langle \cdot\,|\cdot\rangle_{\partial\Omega}$, $\langle \cdot\,|\cdot\rangle_{\Sigma_1}$, and $\langle \cdot\,|\cdot\rangle_{\Sigma_2}$. In the duality pairing the superscript $\Sigma_i,\ i=\{1,2\}$ for the boundary operator $\gamma_j^{\Sigma_i},\ j=\{1,2\}$ will be omitted, as the integration domain is already clear from the previous convention.

Hence, for our main concern, the abstract integration by parts formula (1) of Assumption 1 can be usefully rewritten as

$$\langle Le_1,\, e_2\rangle_\Omega - \langle \gamma_1 e_1\,|\gamma_2 e_2\rangle_{\Sigma_1} = \left\langle e_1,\, L^\dagger e_2\right\rangle_\Omega + \langle \gamma_1 e_1\,|\gamma_2 e_2\rangle_{\Sigma_2}. \tag{9}$$

Coming back to our example with $L = \operatorname{grad}$, we would take $V_{\partial,i} = H^{\frac{1}{2}}(\Sigma_i),\ i\in\{1,2\}$, allowing to rewrite Green's formula as

$$\int_\Omega \operatorname{grad} u\cdot \mathbf{v} - \langle \gamma_0 u\,|\gamma_n \mathbf{v}\rangle_{H^{\frac{1}{2}}(\Sigma_1),H^{-\frac{1}{2}}(\Sigma_1)} = -\int_\Omega u \operatorname{div}\mathbf{v} + \langle \gamma_0 u\,|\gamma_n \mathbf{v}\rangle_{H^{\frac{1}{2}}(\Sigma_2),H^{-\frac{1}{2}}(\Sigma_2)}.$$

2.3 Abstract linear port-Hamiltonian systems

Several linear port-Hamiltonian systems, in particular wave-like hyperbolic systems can then be expressed by means of the abstract dynamical systems

$$\begin{bmatrix} Q_1 & 0\\ 0 & Q_2\end{bmatrix} \frac{\partial}{\partial t}\begin{pmatrix} e_1\\ e_2\end{pmatrix} = \begin{bmatrix} 0 & -L^\dagger\\ L & 0\end{bmatrix}\begin{pmatrix} e_1\\ e_2\end{pmatrix},$$

in terms of the efforts (or co-energy) variables, with Hamiltonian

$$H = \frac{1}{2}\langle e_1,\, Q_1 e_1\rangle_{L^2(\Omega,\mathbb{A})} + \frac{1}{2}\langle e_2,\, Q_2 e_2\rangle_{L^2(\Omega,\mathbb{B})}.$$

The operators $Q_1,\ Q_2$ are bounded algebraic operators, symmetric and positive definite. The boundary conditions are expressed by means of boundary control inputs using G_u defined by (7)

$$\begin{pmatrix} u_1\\ u_2\end{pmatrix} = G_u \begin{pmatrix} e_1\\ e_2\end{pmatrix}. \tag{10}$$

The collocated outputs are then expressed via

$$\begin{pmatrix} y_1\\ y_2\end{pmatrix} = \begin{bmatrix} 0 & \gamma_2^{\Sigma_1}\\ \gamma_1^{\Sigma_2} & 0\end{bmatrix}\begin{pmatrix} e_1\\ e_2\end{pmatrix} = G_y\begin{pmatrix} e_1\\ e_2\end{pmatrix},$$

where $G_y \in \mathcal{L}(D(L)\times D(L^\dagger), \mathcal{Y}_1\times\mathcal{Y}_2)$.

Remark 2. The theoretical construction of G_y starting from the operators L and G_u is far to be trivial, see Brugnoli et al. (2022), but easily identifiable as soon as the Green's formula associated to L, γ_1 and γ_2 is known. For our example $L = \operatorname{grad}$ with Dirichlet and normal traces, it directly leads to

$$G_y = \begin{bmatrix} 0 & \gamma_n^{\Sigma_1}\\ \gamma_0^{\Sigma_2} & 0\end{bmatrix},$$

i.e. observation on Σ_1 is given by the normal trace while observation on Σ_2 is given by the Dirichlet trace.

Thanks to the integration by parts formula (9), it is immediate to verify that

$$\dot H = \langle u_1\,|y_1\rangle_{\mathcal{U}_1,\mathcal{Y}_1} + \langle y_2\,|u_2\rangle_{\mathcal{Y}_2,\mathcal{U}_2}. \tag{11}$$

2.4 Weak formulation and generalized Hellinger-Reissner principle

We first introduce a classical weak formulation of the problem, obtained by taking the inner product with the test functions $v_1,\ v_2$

$$\begin{aligned} \langle v_1,\, Q_1\partial_t e_1\rangle_\Omega &= -\left\langle v_1,\, L^\dagger e_2\right\rangle_\Omega,\\ \langle v_2,\, Q_2\partial_t e_2\rangle_\Omega &= +\left\langle v_2,\, Le_1\right\rangle_\Omega. \end{aligned} \tag{12}$$

A completely analogous formulation is obtained by summing a zero contribution term to both lines of the system. In particular from Eq. (10), it holds

$$u_1 - \gamma_1^{\Sigma_1} e_1 = 0, \qquad u_2 - \gamma_2^{\Sigma_2} e_2 = 0.$$

Taking the duality product of these expressions with the test functions $v_1,\ v_2$ leads to a modified weak formulation

$$\begin{aligned} \langle v_1,\, Q_1\partial_t e_1\rangle_\Omega &= -\left\langle v_1,\, L^\dagger e_2\right\rangle_\Omega + \langle \gamma_1 v_1\,|u_2 - \gamma_2 e_2\rangle_{\Sigma_2},\\ \langle v_2,\, Q_2\partial_t e_2\rangle_\Omega &= +\left\langle v_2,\, Le_1\right\rangle_\Omega + \langle u_1 - \gamma_1 e_1\,|\gamma_2 v_2\rangle_{\Sigma_1}. \end{aligned}$$

Now the system can be put into weak form by performing an integration by parts on either line of the system.

2.4.1. Integration by parts of the $L^\dagger$ operator From the integration by parts formula (9), if the first line is integrated by parts, the first weak formulation is obtained: find $e_1\in D(L),\ e_2\in D(L^\dagger)$ such that

$$\begin{aligned} \langle v_1,\, Q_1\partial_t e_1\rangle_\Omega &= -\left\langle Lv_1,\, e_2\right\rangle_\Omega + \langle \gamma_1 v_1\,|\gamma_2 e_2\rangle_{\Sigma_1}\\ &\quad + \langle \gamma_1 v_1\,|u_2\rangle_{\Sigma_2},\\ \langle v_2,\, Q_2\partial_t e_2\rangle_\Omega &= +\left\langle v_2,\, Le_1\right\rangle_\Omega - \langle \gamma_1 e_1\,|\gamma_2 v_2\rangle_{\Sigma_1}\\ &\quad + \langle u_1\,|\gamma_2 v_2\rangle_{\Sigma_1}. \end{aligned} \tag{13}$$

$\forall v_1\in D(L),\ \forall v_2\in D(L^\dagger)$. The test functions do not carry any information concerning the boundary conditions as those are incorporated in a completely weak manner. The bilinear form

$$\begin{aligned} j_{L,\Sigma_1}((v_1,v_2),(e_1,e_2)) &= -\left\langle Lv_1,\, e_2\right\rangle_\Omega + \langle \gamma_1 v_1\,|\gamma_2 e_2\rangle_{\Sigma_1}\\ &\quad + \left\langle v_2,\, Le_1\right\rangle_\Omega - \langle \gamma_1 e_1\,|\gamma_2 v_2\rangle_{\Sigma_1}, \end{aligned} \tag{14}$$

is skew symmetric.

2.4.2. Integration by parts of the L operator If the second line is integrated by parts, the second weak formulation is obtained: find $e_1\in D(L),\ e_2\in D(L^\dagger)$ such that

$$\begin{aligned} \langle v_1,\, Q_1\partial_t e_1\rangle_\Omega &= -\left\langle v_1,\, L^\dagger e_2\right\rangle_\Omega - \langle \gamma_1 v_1\,|\gamma_2 e_2\rangle_{\Sigma_2}\\ &\quad + \langle \gamma_1 v_1\,|u_2\rangle_{\Sigma_2},\\ \langle v_2,\, Q_2\partial_t e_2\rangle_\Omega &= +\left\langle L^\dagger v_2,\, e_1\right\rangle_\Omega + \langle \gamma_1 e_1\,|\gamma_2 v_2\rangle_{\Sigma_2}\\ &\quad + \langle u_1\,|\gamma_2 v_2\rangle_{\Sigma_1}, \end{aligned} \tag{15}$$

$\forall v_1\in D(L),\ \forall v_2\in D(L^\dagger)$. The bilinear form

$$\begin{aligned} j_{L^\dagger,\Sigma_2}((v_1,v_2),(e_1,e_2)) &= -\left\langle v_1,\, L^\dagger e_2\right\rangle_\Omega - \langle \gamma_1 v_1\,|\gamma_2 e_2\rangle_{\Sigma_2}\\ &\quad + \left\langle L^\dagger v_2,\, e_1\right\rangle_\Omega + \langle \gamma_1 e_1\,|\gamma_2 v_2\rangle_{\Sigma_2}, \end{aligned} \tag{16}$$

is clearly skew symmetric.

Remark 3. Since $v_1,\ e_1\in D(L)$ and $v_2,\ e_2\in D(L^\dagger)$, by using the integration by parts (9) on the appropriate line of the bilinear forms j_{L,Σ_1} or $j_{L^\dagger,\Sigma_2}$, we obtain that $j_{L,\Sigma_1} = j_{L^\dagger,\Sigma_2}$.

Remark 4. In the mixed finite element method, the variable that is not subject to differentiation can be chosen less regular, i.e. L^2. Here however, since both the bilinear forms (14) and (16) contain a boundary duality product, all the variables need to be regular enough. This consideration leads to the weak formulations (13), (15) where $v_1, e_1 \in D(L)$ and $v_2, e_2 \in D(L^\dagger)$.

2.4.3. Links with Lagrange multipliers Let us consider the case of the boundary condition on Σ_1 imposed by the Lagrange multiplier method. The idea is to extend the system by an extra variable λ, namely the Lagrange multiplier associated to the constraint $u_1 - \gamma_1^{\Sigma_1} e_1 = 0$. Using integration by parts (1) on the first line of (12), we get the extended system in weak form: find $e_1 \in D(L)$, $e_2 \in D(L^\dagger)$, $\lambda \in \gamma_1^{\Sigma_1}(D(L))$ such that

$$
\begin{aligned}
\langle v_1,\, Q_1 \partial_t e_1 \rangle_\Omega &= -\langle L v_1,\, e_2 \rangle_\Omega + \langle \gamma_1 v_1 \,|\lambda \rangle_{\Sigma_1} + \langle \gamma_1 v_1 \,| u_2 \rangle_{\Sigma_2}, \\
\langle v_2,\, Q_2 \partial_t e_2 \rangle_\Omega &= +\langle v_2,\, L e_1 \rangle_\Omega, \\
0 &= \langle u_1 - \gamma_1 e_1 \,| v_\lambda \rangle_{\Sigma_1},
\end{aligned}
$$

In Brugnoli et al. (2020), it has been shown that $\lambda = y_1 := \gamma_2^{\Sigma_1} e_2$ in this case. Hence, assuming $v_\lambda = \gamma_2 v_2$ and substituting the third line in the second one leads to (13).

The same holds true in the other way: imposing the boundary condition on Σ_2 leads to $\lambda := \gamma_1^{\Sigma_2} e_1$. Then integrating by part the second line in (12) and substituting the constraint in the first one leads to (15), assuming $v_\lambda = \gamma_1 v_1$.

2.5 Finite-dimensional systems

Introducing the finite element expansion for the test functions efforts and control inputs

$$
v_i = \sum_{m=1}^{N_i} \phi_i^m(\boldsymbol{x}) v_i^m, \qquad e_i = \sum_{m=1}^{N_i} \phi_i^m(\boldsymbol{x}) e_i^m(t), \qquad \boldsymbol{x} \in \Omega,
$$

$$
u_i = \sum_{m=1}^{N_{i,\partial}} \psi_i^m(\boldsymbol{s}_i) u_i^m(t), \qquad \boldsymbol{s}_i \in \Sigma_i \qquad i = \{1, 2\},
$$

the following finite-dimensional system is obtained from the weak formulation (13)

$$
\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_L^\top \\ \mathbf{D}_L & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B}_2 \\ \mathbf{B}_1 & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}. \tag{17}
$$

The mass matrix is constructed via

$$
[\mathbf{M}_i]_{mn} = \langle \phi_i^m,\, Q_i \phi_i^n \rangle_\Omega,
$$

the differentiation matrix is given by

$$
[\mathbf{D}_L]_{mn} = \langle \phi_2^m,\, L \phi_1^n \rangle_\Omega - \langle \gamma_1 \phi_1^n \,| \gamma_2 \phi_2^m \rangle_{\Sigma_1},
$$

the control matrices are computed via

$$
[\mathbf{B}_1]_{mn} = \langle \psi_1^n \,| \gamma_2 \phi_2^m \rangle_{\Sigma_1}, \qquad [\mathbf{B}_2]_{mn} = \langle \gamma_1 \phi_1^m \,| \psi_2^n \rangle_{\Sigma_2}.
$$

Symmetrically, starting from the weak formulation (15), the following system is readily obtained

$$
\begin{bmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2 \end{bmatrix} \begin{pmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{L^\dagger} \\ \mathbf{D}_{L^\dagger}^\top & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{B}_2 \\ \mathbf{B}_1 & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}, \tag{18}
$$

where the differentiation matrix $\mathbf{D}_{L^\dagger}$ now reads

$$
[\mathbf{D}_{L^\dagger}]_{mn} = \left\langle \phi_1^m,\, L^\dagger \phi_2^n \right\rangle_\Omega + \langle \gamma_1 \phi_1^m \,| \gamma_2 \phi_2^n \rangle_{\Sigma_2}.
$$

It is worth noting that $\mathbf{D}_L$ is exactly the discretization of the left-hand side of (9), while $\mathbf{D}_{L^\dagger}$ corresponds to its right-hand side. In particular, for conforming discrete spaces

$$
\begin{aligned}
\mathrm{span}(\phi_1^1, \dots, \phi_1^{N_1}) &= V_L \subset D(L), \\
\mathrm{span}(\phi_2^1, \dots, \phi_2^{N_2}) &= V_{L^\dagger} \subset D(L^\dagger),
\end{aligned}
$$

the Stokes theorem (9) leads to the algebraic identity $\mathbf{D}_L^\top = \mathbf{D}_{L^\dagger}$.

3. SOME ENGINEERING EXAMPLES

3.1 Wave equation

The wave equation in an bounded domain $\Omega \subset \mathbb{R}^d$ is described by the following system

$$
\begin{bmatrix} \kappa^{-1} & 0 \\ 0 & \rho \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} p \\ \boldsymbol{u} \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & 0 \end{bmatrix} \begin{pmatrix} p \\ \boldsymbol{u} \end{pmatrix}, \tag{19}
$$

where the unknowns are the pressure scalar field $p : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{R}$ and the velocity vector field $\boldsymbol{u} : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{V}$. The physical parameters are the bulk modulus $\kappa : \Omega \to \mathbb{R}$ and the mass density $\rho : \Omega \to \mathbb{R}$. These parameters are considered time independent. For this model the L operator and its adjoint $L^\dagger$ corresponds to the gradient and minus the vector divergence respectively

$$
L = \mathrm{grad}, \qquad L^\dagger = -\,\mathrm{div},
$$

with domains given by Eqs. (2) and (5) respectively. The integration by parts formula (6) then holds, with the trace operator γ_1 given by the Dirichlet trace (3) and γ_2 represented by the normal trace (4).

3.2 Elastodynamics

The linear elastodynamics problem in $\Omega \subset \mathbb{R}^d$ is expressed by the following system

$$
\begin{bmatrix} \rho & 0 \\ 0 & \boldsymbol{\mathcal{C}} \end{bmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\Sigma} \end{pmatrix} = \begin{bmatrix} 0 & \mathrm{Div} \\ \mathrm{Grad} & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{\Sigma} \end{pmatrix}.
$$

The unknowns are the velocity field $\boldsymbol{u} : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{V}$ and the symmetric stress tensor $\boldsymbol{\Sigma} : \Omega \times (0, t_{\mathrm{end}}) \to \mathbb{S}$. The physical parameters correspond to the density $\rho : \Omega \to \mathbb{R}$ and the compliance fourth order tensor $\boldsymbol{\mathcal{C}} : \Omega \to \mathcal{L}(\mathbb{S})$. The operator L and its adjoint with respect to G_u are given by

$$
L = \mathrm{Grad}, \qquad L^\dagger = -\,\mathrm{Div},
$$

where $\mathrm{Grad}\,\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla^\top \boldsymbol{u})$ is the symmetric gradient of vector fields and $\mathrm{Div}\,\boldsymbol{\Sigma} = \sum_{i=1}^d \partial_{x_i} [\boldsymbol{\Sigma}]_{ij}$ is the columnwise divergence of tensor fields. The operators domains are given by

$$
D(\mathrm{Grad}) = H^{\mathrm{Grad}}(\Omega, \mathbb{V}), \qquad D(\mathrm{Div}) = H^{\mathrm{Div}}(\Omega, \mathbb{S}),
$$

where the following Sobolev spaces have been introduced

$$
\begin{aligned}
H^{\mathrm{Grad}}(\Omega, \mathbb{V}) &:= \{\boldsymbol{u} \in L^2(\Omega, \mathbb{V}) |\ \mathrm{Grad}\,\boldsymbol{u} \in L^2(\Omega, \mathbb{S})\}, \\
H^{\mathrm{Div}}(\Omega, \mathbb{S}) &:= \{\boldsymbol{\Sigma} \in L^2(\Omega, \mathbb{S}) |\ \mathrm{Div}\,\boldsymbol{\Sigma} \in L^2(\Omega, \mathbb{V})\}.
\end{aligned}
$$

The following integration by parts formula then holds

$$
\langle \mathrm{Grad}\,\boldsymbol{u},\, \Sigma \rangle_\Omega + \langle \boldsymbol{u},\, \mathrm{Div}\,\Sigma \rangle_\Omega = \langle \gamma_1 \boldsymbol{u} \,| \gamma_2 \boldsymbol{\Sigma} \rangle_{\partial\Omega},
$$

where the trace operators corresponds to the vector Dirichlet trace and to the normal trace of tensors

$$
\gamma_1 \boldsymbol{u} = \boldsymbol{u}|_{\partial\Omega}, \qquad \gamma_2 \boldsymbol{\Sigma} = \boldsymbol{\Sigma} \cdot \boldsymbol{n}.
$$

The boundary duality product involve the spaces

$$
\begin{aligned}
V_\partial &= H^{1/2}(\partial\Omega, \mathbb{V}) := \mathrm{tr}\, H^{\mathrm{Grad}}(\Omega, \mathbb{V}), \\
V_\partial' &= H^{-1/2}(\partial\Omega, \mathbb{V}),
\end{aligned}
$$

where $H^{1/2}(\partial\Omega, \mathbb{V})$ corresponds to the range of the trace operator and $H^{-1/2}(\partial\Omega, \mathbb{V})$ is its topological dual.

4. NUMERICAL RESULTS AND DISCUSSION

4.1 An eigenvalue problem for the 2D wave equation

We consider an eigenvalue problem for the wave equation (19) with unitary physical parameters, $\kappa = 1$, $\rho = 1$, in a two-dimensional rectangular domain

$$\Omega = \{(x, y) \in [0, \pi] \times [0, \pi]\},$$

together with the boundary partition

$$\Sigma_1 = \{x = 0 \cup x = \pi\}, \qquad \Sigma_2 = \{y = 0 \cup y = \pi\}.$$

This means that Dirichlet homogeneous boundary conditions are imposed on the left and right sides

$$p|_{x=0} = 0, \qquad p|_{x=\pi} = 0,$$

whereas Neumann boundary conditions are imposed on the lower and upper sides

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{y=0} = 0, \qquad \boldsymbol{u} \cdot \boldsymbol{n}|_{y=\pi} = 0.$$

For this problem the analytical eigenvalues take the form

$$\begin{aligned} \lambda_{\text{ex}} &= \pm j\omega_{\text{ex}}, \\ \omega_{\text{ex}} &= \sqrt{n^2 + m^2}, \qquad \forall n \in \mathbb{N}_0,\ \forall m \in \mathbb{N}_{>0} \end{aligned}$$

where $j = \sqrt{-1}$ is the imaginary unit. For the discretization, the grad − grad formulation is considered. The weak imposition of the boundary conditions is compared against a standard strong imposition of the Dirichlet boundary condition. The finite-dimensional system (17) is employed for the former approach, leading to the following eigenproblem

$$j\omega_i \begin{bmatrix} \mathbf{M}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_u \end{bmatrix} \begin{pmatrix} \boldsymbol{\psi}_p^i \\ \boldsymbol{\psi}_u^i \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{D}_{\text{grad}}^\top \\ \mathbf{D}_{\text{grad}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\psi}_p^i \\ \boldsymbol{\psi}_u^i \end{pmatrix}. \tag{20}$$

For the strong imposition case, the following system is obtained instead

$$j\omega_i \begin{bmatrix} \mathbf{M}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_u \end{bmatrix} \begin{pmatrix} \boldsymbol{\psi}_p^i \\ \boldsymbol{\psi}_u^i \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -([\mathbf{K}_{\text{grad}}]^{\Sigma_1})^\top \\ [\mathbf{K}_{\text{grad}}]^{\Sigma_1} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \boldsymbol{\psi}_p^i \\ \boldsymbol{\psi}_u^i \end{pmatrix}, \tag{21}$$

where $\mathbf{K}_{\text{grad}}$ is computed as $[\mathbf{K}_{\text{grad}}]_{mn} = \langle \phi_2^m, \text{grad}\, \phi_1^n \rangle_\Omega$. The notation $[\mathbf{K}_{\text{grad}}]^{\Sigma_1}$ indicates that the columns of matrix $\mathbf{K}_{\text{grad}}$ corresponding to the degrees of freedom on Σ_1 are replaced by the corresponding columns of the identity matrix [1]. This ensures the correct handling of the boundary conditions. Systems (20), (21) are compactly rewritten as

$$\begin{aligned} j\omega_i \mathbf{M}\boldsymbol{\psi}_i &= \mathbf{J}_{\text{weak}}\boldsymbol{\psi}_i, \qquad &\text{for System (20)}, \\ j\omega_i \mathbf{M}\boldsymbol{\psi}_i &= \mathbf{J}_{\text{strong}}\boldsymbol{\psi}_i, \qquad &\text{for System (21)}. \end{aligned}$$

For what concerns the choice of the finite element spaces, Continous Galerkin of degree r are employed for the pressure $p_h \in \text{CG}_r$, while Raviart-Thomas of degree r are used for the velocity $\boldsymbol{u}_h \in \text{RT}_r$. A precise description of the Continuous Galerkin and Raviart-Thomas finite element spaces can be found in Brenner et al. (2008) and Boffi et al. (2013) respectively. It is worth recalling that the scalar field space CG_r is continuous across elements, whereas the space RT_r represents vector fields that have continuous normal component across elements. These two spaces form a de Rham subcomplex $\text{CG}_r \xrightarrow{\text{grad}} \text{RT}_r$. This means that they preserve the cohomology associated to the de Rham complex. This is a core property that is fundamental for the stability and consistency of mixed finite elements Arnold et al. (2006).

[1] The interested reader may consult `https://www.firedrakeproject.org/boundary_conditions.html` for a detailed explanation

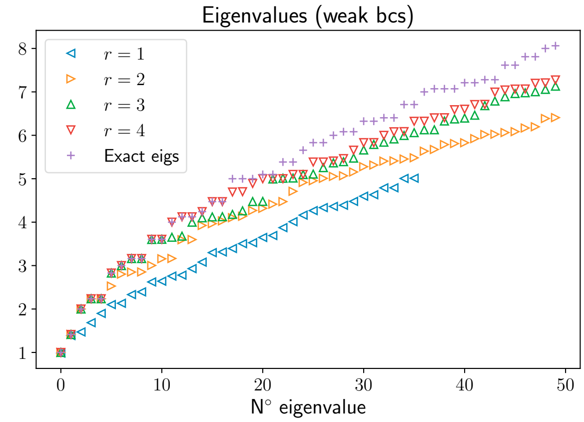

Figure 1. Eigenvalues computed via weak imposition of the boundary conditions

Remark 5. The weak formulations (13), (15) are quite restrictive for the choice of finite elements, since both spaces need to be conforming with respect to the L and $L^\dagger$ operators. In particular it becomes more difficult to select finite element spaces that form a de Rham subcomplex. For example, the div-div discretization of the wave equation RT_r and DG_{r-1} elements can be used, whereas the weak imposition of the boundary conditions demands the employment of RT_r and CG_r finite elements.

4.2 Discussion

The finite element library Firedrake (see Rathgeber et al. (2017)) was used to generate the finite element matrices. For the discretization 5 elements per side are used

$$N_x^{\text{el}} = 5, \qquad N_y^{\text{el}} = 5.$$

This means that the mesh consists of 50 triangles. To compute the eigenvalues, the Krylov-Schur solver from the SLEPc library Hernandez et al. (2005) is employed. A shift and invert spectral transform is used to look for the eigenvalues in the lowest part of the spectrum.

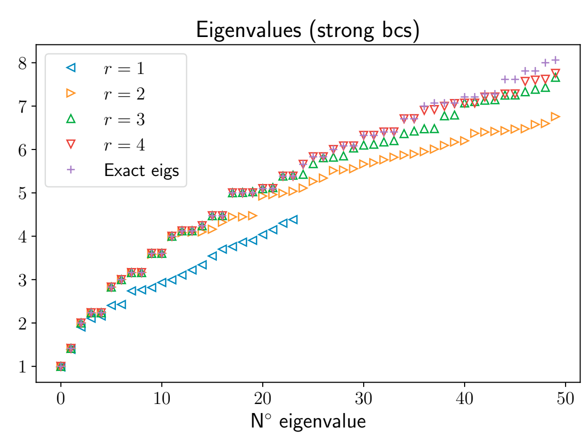

Figure 2. Eigenvalues computed via strong imposition of the boundary conditions

The results for the weak and strong impositions of the eigenvalues are reported in Figs. 1 and 2 respectively. Symbol $+$ denotes the exact eigenvalues, whereas the colored triangles represent the numerical solution for different values of the polynomial degree, $r = \{1, 2, 3, 4\}$. It can be immediately noticed that the weak imposition of the boundary condition leads to poorer results compared to the strong one. For the highest degree $r = 4$ approximately 20 eigenvalues are correctly computed in the case of weak boundary conditions, while more than the double are correctly computed by the strong imposition of the boundary conditions. So even if this approach allows obtaining explicit pH systems, it does not perform as well as the canonical strong imposition of the essential Dirichlet condition. Another point that deserves a deeper analysis is the appearance of zero eigenvalues in the spectrum when considering eigenproblems of pH systems. This is related to the fact that the Dirichlet condition is imposed on the time derivative of the original field. To this end consider the canonical form of the wave equation as a second order system in time and space

$$\kappa^{-1}\partial_{tt}w - \mathrm{div}(\rho^{-1}\,\mathrm{grad}\, w) = 0.$$

System (19) is obtained by introducing the following variables $p = \partial_t w, \qquad \boldsymbol{u} = \rho^{-1}\,\mathrm{grad}\, w$. Setting to zero the p variable leads to $j\omega_i \psi_w^i|_{\Sigma_1} = 0$, leading to additional zeros in the spectrum.

The Maxwell equations in 3D can also be treated, see Haine et al. (2022).

5. CONCLUSION

In this paper, the approach proposed on an example in Thoma and Kotyczka (2021) has been extended to the general case of abstract linear pH systems. The formulation incorporates a boundary duality pairing into the interconnection operator to accommodate for mixed boundary conditions. It has been shown that this lead to two completely equivalent weak formulations and corresponding finite-dimensional systems. Moreover, the Hellinger-Reissner principle can be equivalently obtained by considering the reduction of the constraint associated with a Lagrange multiplier method, assuming a suitable choice of the multiplier discrete basis. This approach allows avoiding the need to deal with differential algebraic systems, but exhibits some serious drawbacks. First of all the choice of the finite elements is restricted to more regular elements. These may not satisfy de Rham subcomplex property (cf. Rmk. 5). Furthermore, the results for the considered test case show that the approach performs rather poorly compared to the standard strong imposition. Future developments will consider different strategies to impose the boundary conditions in a weak manner and extend those to the case of elasticity problems.

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