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Incompressible Navier-Stokes Equation as port-Hamiltonian systems: velocity formulation versus vorticity formulation ⋆

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Abstract: Starting from the description of the isentropic compressible viscous fluid as port-Hamiltonian system in [Mora & al., 2020], the special cases of irrotational or incompressible cases in 2D or 3D are investigated. For the incompressible fluid, the non-linear Navier-Stokes equations are first presented with velocity as energy variable, then analyzed as a modulated port-Hamiltonian system with the help of the vorticity as energy variable. Finally, the structure-preserving numerical scheme provided by the Partitioned Finite Element Method (PFEM) of [Serhani & al., 2019] is applied to the incompressible dissipative fluid in 2D.

Keywords: Navier-Stokes Equations (NSE), Port-Hamiltonian systems (pHs), Incompressible fluid, vorticity, Partitioned Finite Element Method (PFEM)

1. INTRODUCTION

The goal of this work is to present advantageous numerical method to simulate the non-linear Navier-Stokes equations for an incompressible Newtonian fluid, with collocated boundary control and observation.

Accurate mathematical description of fluids can be found in e.g. Chorin and Marsden (1992), Morrison (1998), Boyer and Fabrie (2013).
One of the first references on the port-Hamiltonian fomalism in infinite dimension is van der Schaft and Maschke (2002). Recently a comprehensive focus on fluid mechanics models using differential geometry has been proposed in Califano et al. (2021), based on two more theoretical papers, namely Rashad et al. (2021a,b).
Navier-Stokes in 1D has been tackled by Altmann and Schulze (2017), whereas in higher space dimension, vorticity naturally appears: this has been accounted for in Mora et al. (2020). Useful references on vorticity include the seminal book Truesdell (1954), several parts of Olver (1993), and also Polner and van der Vegt (2014) for the Euler equations, and Castro and Lannes (2015) for the Shallow Water equations.
One example of constraints on a dynamical system is the divergence-free condition for an incompressible fluid, hence the underlying mathematical setting is that of Differential Algrebraic equations, see e.g. Kunkel and Mehrmann (2006), and more specifically port-Hamiltonian Differential Algrebraic equations: we refer to van der Schaft (2013), van der Schaft and Maschke (2018),Beattie et al. (2018) and Brugnoli et al. (2020).
A structure-preserving method used to simulate port-Hamiltonian system is the Partitioned Finite Element Method (PFEM), introduced in Cardoso-Ribeiro et al. (2021) for lossless pHs, extended to passive pHs in Serhani et al. (2019b); many applications have already been dealt with: Cardoso-Ribeiro et al. (2019) for the Shallow Water, Serhani et al. (2019a) for the heat equation, Brugnoli et al. (2019) for plate equations, Payen et al. (2020) for Maxwell's equations. The implementation details of this method can be found in Brugnoli et al. (2021), together with NoteBooks as supplementary material.

In this work, the Incompressible Navier-Stokes Equations (INSE) are investigated as port-Hamiltonian system. The main contributions of this paper are:

(1) the vorticity (ω) formulation, making use of the stream function (ψ) as co-energy variable, and the expression of the collocated boundary controls and observations in terms of ψ .
(2) the application of the structure-preserving method PFEM to the vorticity formulation, where the integration by part has to be performed on the scalar structure operator J_ω, which is differentially modulated by the energy variable ω.

The paper is organized as follows: in § 2 the compressible isentropic fluid is recalled, while the specific case of incompressible fluids is presented in § 3 both in velocity and vorticity formulations. Then in § 4, the PFEM method is applied to the vorticity formulation in 2D. Finally conclusions are drawn and perspective suggested in § 5.

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2. COMPRESSIBLE FLUIDS

The objective is to recall how the general Navier-Stokes equations with collocated boundary control and observations can be recast into the framework of port-Hamiltonian systems with dissipation for an isentropic Newtonian fluid, following (Mora et al., 2020, § 3).

Following e.g. (Boyer and Fabrie, 2013, chap. 1), the conservation of mass reads:

$$\partial_t \rho + \operatorname{div}(\rho\, \boldsymbol{u}) = 0\,, \tag{1}$$

and the evolution of the linear momentum reads, in non-conservative form:

$$\rho\, D_t \boldsymbol{u} := \rho\,(\partial_t + \boldsymbol{u}\cdot\mathbf{grad})\boldsymbol{u} \tag{2}$$

$$= -\,\mathbf{grad}(P) + \mu\,\boldsymbol{\Delta u} + (\lambda+\mu)\,\mathbf{grad}(\operatorname{div}(\boldsymbol{u})) \tag{3}$$

The positive coefficients are μ the dynamic viscosity, and $\lambda + \frac{2}{3}\mu$ the bulk viscosity, the latter being equal to 0 under Stokes assumption (in which case, $\lambda = -\frac{2}{3}\mu$).

Together with appropriate boundary variables, the previous equations of the compressible Newtonian fluid are recast as pHs in the rotational case in § 2.1, and in the irrotational case in § 2.2.

2.1 General case: compressible and rotational

Let $\rho \mapsto e(\rho)$ the internal energy density, and define the Hamiltonian functional as $\mathcal{H} := \int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2 + \rho\, e(\rho)$. Choosing as energy variables density ρ, and velocity $\boldsymbol{u}$, and using the canonical scalar product, one can compute the co-energy variables $e_\rho := \delta_\rho \mathcal{H} = \frac{1}{2}|\boldsymbol{u}|^2 + \frac{P}{\rho} = h(\rho, \boldsymbol{u})$ the enthalpy, and $\boldsymbol{e_u} := \delta_{\boldsymbol{u}}\mathcal{H} = \rho\boldsymbol{u}$ the linear momentum density.

Then, making use of vector calculus identities, (A.1) to take the flow into account, and (B.3) to factorize the *vectorial* Laplacian out, we introduce two extra algebraic ports, curly and divergent,

- the skew-symmetric matrix $G(\boldsymbol{\omega}) = \boldsymbol{\omega}\wedge$ responsible for the gyroscopic term,
- $\boldsymbol{f}_c := \boldsymbol{\omega} = \mathbf{curl}\,\boldsymbol{u} = \mathbf{curl}(\rho^{-1}\boldsymbol{e_u})$,
- $f_d := \operatorname{div}\boldsymbol{u} = \operatorname{div}(\rho^{-1}\boldsymbol{e_u})$,

which are physically meaningful, and add the **closure relation** $e_d = \mu_d\, f_d$ and $\boldsymbol{e}_c = \mu_c\, \boldsymbol{f}_c$ (with $\mu_c = \mu$ and $\mu_d = \lambda + 2\mu = \frac{4}{3}\mu$). Then, we are in a position to recast (1)-(2) as follows:

$$\begin{pmatrix} \partial_t\rho \\ \partial_t \boldsymbol{u} \\ \boldsymbol{f}_c \\ f_d \end{pmatrix} = \mathcal{J}_e \begin{pmatrix} e_\rho \\ \boldsymbol{e_u} \\ \boldsymbol{e}_c \\ e_d \end{pmatrix}, \tag{4}$$

where the interconnection differential operator $\mathcal{J}_e$ is:

$$\mathcal{J}_e = \begin{bmatrix} 0 & -\operatorname{div} & 0 & 0 \\ -\mathbf{grad} & -\rho^{-1}.G(\boldsymbol{\omega}) & -\rho^{-1}.\mathbf{curl} & \rho^{-1}.\mathbf{grad} \\ 0 & \mathbf{curl}(\rho^{-1}.) & 0 & 0 \\ 0 & \operatorname{div}(\rho^{-1}.) & 0 & 0 \end{bmatrix}, \tag{5}$$

the constitutive relations can be collected into vectors $\boldsymbol{e}_r$ and $\boldsymbol{f}_r$ and the closure relation reads $\boldsymbol{e}_r = S\boldsymbol{f}_r$, with $S := \operatorname{diag}(\mu_c I_3, \mu_d)$.

For the power balance, a technical computation gives:

Theorem 1. The evolution of the Hamiltonian along the trajectories of the dynamical system (4) with the closure relations is given by:

$$\frac{d}{dt}\mathcal{H}(t) = -\int_\Omega \mu_d\, {f_d}^2 - \int_\Omega \mu_c\,|\boldsymbol{f_c}|^2$$

$$+ \int_{\partial\Omega} [-e_\rho\, \boldsymbol{e_u}\cdot\boldsymbol{n} + \mu_d\, \operatorname{div}(\boldsymbol{u})\, \frac{\boldsymbol{e_u}}{\rho}\cdot\boldsymbol{n} + \mu_c\, \boldsymbol{\omega}\cdot(\frac{\boldsymbol{e_u}}{\rho}\wedge\boldsymbol{n})] \tag{6}$$

which is physically meaningful: both the normal and the tangential components of the velocity vector at the boundary play a role.
A couple of possible boundary controls is $u_n := \boldsymbol{u}\cdot\boldsymbol{n}$ and $\boldsymbol{u_\tau} := \boldsymbol{u}\wedge\boldsymbol{n}$, with collocated boundary observations $y_n := -\rho\,(-e_\rho + \mu_d\, \operatorname{div}(\boldsymbol{u}))$ and $\boldsymbol{y_\tau} := \mu_c\,\rho\,\boldsymbol{\omega}$.

Proof. The result is available in an equivalent form in (Mora et al., 2020, § 3), however the proof below makes appear the two physically meaningful components explicitely, namely $\operatorname{div}(\boldsymbol{u})$ and $\boldsymbol{\omega} = \mathbf{curl}(\boldsymbol{u})$.

$$(\boldsymbol{e}, \mathcal{J}_e\boldsymbol{e}) = \int_\Omega e_\rho\,\partial_t\rho + \boldsymbol{e_u}\cdot\partial_t\boldsymbol{u} + \boldsymbol{e_r}\cdot\boldsymbol{f_r}$$

$$= \int_{\partial\Omega} [-e_\rho\, \boldsymbol{e_u}\cdot\boldsymbol{n} + \mu_d\, \operatorname{div}(\boldsymbol{u})\, \boldsymbol{e_u}\cdot\boldsymbol{n} + \mu_c\, \mathbf{curl}(\boldsymbol{u})\cdot(\boldsymbol{e_u}\wedge\boldsymbol{n})]$$

In the above computation, the Stokes formula with the **curl** (A.4) has been used, as is usual in electromagnetics when dealing with Maxwell's equations, see e.g. Payen et al. (2020). □

Since numerics will be addressed in the sequel, it becomes obvious that all the terms involving ρ^{-1} will be difficult to tackle, that is the reason why an alternative formulation can be chosen, based on another inner product, used to compute the variational derivatives, namely L^2_ρ. Indeed, defining $\boldsymbol{e}_u := \delta^\rho_{\boldsymbol{u}}\mathcal{H} = \boldsymbol{u}$, in L^2_ρ, and multiplying the second line by ρ, the following system is readily obtained:

$$\begin{pmatrix} \partial_t\rho \\ \rho\,\partial_t\boldsymbol{u} \\ \boldsymbol{f}_c \\ f_d \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div}(\rho\,.) & 0 & 0 \\ -\rho\,\mathbf{grad} & G(\boldsymbol{\omega}) & -\mathbf{curl} & \mathbf{grad} \\ 0 & \mathbf{curl} & 0 & 0 \\ 0 & \operatorname{div} & 0 & 0 \end{bmatrix} \begin{pmatrix} e_\rho \\ \boldsymbol{e_u} \\ \boldsymbol{e}_c \\ e_d \end{pmatrix}. \tag{7}$$

This latter formulation will definitely simplify the application of PFEM. Even if this will require the same number of matrices to update in the interconnection operator:

(1) $\rho^{-1}G(\boldsymbol{\omega}), -\rho^{-1}\,\mathbf{curl}$ and $\rho^{-1}\,\mathbf{grad}$,

(2) $\rho\,.$ (on the left-hand side), $-\rho\,\mathbf{grad}$ and $G(\boldsymbol{\omega})$,

but it should help to compute the constitutive relations: for e_ρ since we directly have access to $\boldsymbol{e}_u = \boldsymbol{u}$ instead of $\partial_t\boldsymbol{u}$ and $\rho\boldsymbol{u}$ and obviously for $\boldsymbol{e}_u = \boldsymbol{u}$.

2.2 Compressible and Irrotational case

In this case $\boldsymbol{\omega} = \boldsymbol{0}$, thus $G(\boldsymbol{\omega}) = 0$, and the non-linear advection term happily simplifies into a purely gradient term, that is $(\boldsymbol{u}\cdot\mathbf{grad})\boldsymbol{u} = \mathbf{grad}(\frac{1}{2}|\boldsymbol{u}|^2)$, see Appendix A. Hence, the irrotational case appears as a special case of (7), as follows:

$$\begin{pmatrix} \partial_t\rho \\ \rho\partial_t\boldsymbol{u} \\ f_d \end{pmatrix} = \begin{bmatrix} 0 & -\operatorname{div}\rho & 0 \\ -\rho\,\mathbf{grad} & 0 & \mathbf{grad} \\ 0 & \operatorname{div} & 0 \end{bmatrix} \begin{pmatrix} e_\rho \\ \boldsymbol{e_u} \\ e_d \end{pmatrix}. \tag{8}$$

Note that if we add the incompressibility condition, then both conditions imply that $\boldsymbol{\Delta u} = \mathbf{0}$, and the inviscid fluid is recovered as model, which is not our purpose here.

3. INCOMPRESSIBLE FLUIDS

However, defining the incompressible case properly from the general case proves less straightforward, because the pressure term in the enthalpy is lost. Indeed, in this setting, $\rho = \rho_0$ and, up to a constant, the Hamiltonian now reduces to the kinetic part only. Indeed, $\mathcal{H} := \frac{1}{2}\int_\Omega \rho_0 |\boldsymbol{u}|^2$.

In § 3.1, the velocity formulation derived from (7) is presented, and in § 3.2, another choice of energy variables is made, namely vorticity: the new system is then fully analyzed.

3.1 Velocity formulation

With the choice of velocity as energy variable, the incompressible equations read:

$$\begin{pmatrix} \rho_0 \partial_t \boldsymbol{u} \\ \boldsymbol{f}_c \\ 0 \end{pmatrix} = \begin{bmatrix} G(\boldsymbol{\omega}) & -\mathbf{curl} & \mathbf{grad} \\ \mathbf{curl} & 0 & 0 \\ \mathrm{div} & 0 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{e_u} \\ \boldsymbol{e}_c \\ e_d \end{pmatrix}, \qquad (9)$$

In this system, the divergence-free constraint $0 = f_d$ is ensured by the presence of a Lagrange multiplier in the dynamics, under the form $\mathbf{grad}(e_d)$, where $-e_d = p + \frac{1}{2}\rho_0|\boldsymbol{u}|^2$ is the total pressure, that does not possess any thermodynamic meaning. Hence, the pressure is determined up to a constant in these equations. It is intrinsically a port-Hamiltonian Differential Algebraic infinite-dimensional system.

For the power balance, a technical computation gives:

Theorem 2. The evolution of the Hamiltonian along the trajectories of the dynamical system (9) with the closure relation is given by:

$$\frac{d}{dt}\mathcal{H}(t) = -\int_\Omega \mu_c |\boldsymbol{f_c}|^2 + \int_{\partial\Omega} e_d \boldsymbol{u}\cdot\boldsymbol{n} + \mu_c \boldsymbol{\omega}\cdot(\boldsymbol{u}\wedge\boldsymbol{n}). \qquad (10)$$

3.2 Fully nonlinear NSE using of the (vorticity, stream function) description

From now on, we only consider the 2D case. Following (Chorin and Marsden, 1992, §. 1.2), we recall that the curl_{2D} differential operator is defined by $\mathrm{curl}_{2D}(\boldsymbol{v}) := \partial_x v_2 - \partial_y v_1$, and that the following integration by parts formula holds:

$$\int_\Omega \mathrm{curl}_{2D}(\boldsymbol{v})\, w \,\mathrm{d}x = \int_\Omega \boldsymbol{v}\cdot\mathbf{grad}^\perp(w)\,\mathrm{d}x + \int_{\partial\Omega} (\mathcal{R}\boldsymbol{v})\cdot\boldsymbol{n}\, w \,\mathrm{d}s, \qquad (11)$$

where [1] $\mathbf{grad}^\perp(w) := \begin{pmatrix} \partial_y w \\ -\partial_x w \end{pmatrix}$, and $\mathcal{R}$ denotes the rotation of angle $-\frac{\pi}{2}$ in the plane.

[1] Care must be taken that in some references, like Olver (1993) or Morrison (1998), the opposite definition for $\mathbf{grad}^\perp$ is chosen. We stick to this one in order to be consistent with the adjoint of the curl_{2D} operator.

Applying curl_{2D} to the linear momentum conservation equation leads to the following evolution equation for the vorticity $\omega := \mathrm{curl}_{2D}(\boldsymbol{u})$:

$$\rho_0\, \partial_t \omega = \mathrm{curl}_{2D}\left(G(\omega)\ \boldsymbol{u}\right) - \mu_c\, \mathrm{curl}_{2D}\ \mathbf{grad}^\perp(\omega),$$

where $\mathrm{curl}_{2D}\mathbf{grad} \equiv 0$ has been used: this trick eliminates the total pressure from the system.

Assume that the velocity $\boldsymbol{u}$ is fully determined by a stream function ψ, that is a potential such that $\boldsymbol{u} = \mathbf{grad}^\perp\psi := \begin{pmatrix} \partial_y\psi \\ -\partial_x\psi \end{pmatrix}$. Substituting $\boldsymbol{u}$ with this definition gives:

$$\rho_0\, \partial_t\omega = \mathrm{curl}_{2D}\left(G(\omega)\ \mathbf{grad}^\perp(\psi)\right) - \mu_c\, \mathrm{curl}_{2D}\ \mathbf{grad}^\perp(\omega).$$

Proposition 3. For all sufficiently smooth functions ψ:

$$\begin{aligned} \mathrm{curl}_{2D}\left(G(\omega)\ \mathbf{grad}^\perp(\psi)\right) &= \partial_x(\omega\partial_y\psi) - \partial_y(\omega\partial_x\psi), \\ &= \mathrm{div}\left(\omega\ \mathbf{grad}^\perp(\psi)\right), \\ &=: J_\omega\psi. \end{aligned}$$

Furthermore, J_ω is formally skew-symmetric, and satisfies Jacobi identities (see e.g. (Olver, 1993, Example 7.10)).

Proof: Let us compute

$$G(\omega)\mathbf{grad}^\perp(\psi) = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \wedge \begin{pmatrix} \partial_y\psi \\ -\partial_x\psi \\ 0 \end{pmatrix} = \begin{pmatrix} \omega\partial_x\psi \\ \omega\partial_y\psi \end{pmatrix}.$$

Applying curl_{2D} gives the claimed result.

The formal skew-symmetry is then obvious by integration by parts since, for all $\psi \in \mathcal{C}_0^\infty(\Omega)$:

$$\int_\Omega \mathrm{div}\left(\omega\ \mathbf{grad}^\perp(\psi)\right)\psi\ \mathrm{d}x = -\int_\Omega \omega\ \underbrace{\mathbf{grad}^\perp(\psi)\cdot\mathbf{grad}(\psi)}_{=0}\ \mathrm{d}x$$

□

For the power balance, the computation using (11) gives:

Theorem 4. The evolution of the Hamiltonian $\mathcal{H} := \frac{1}{2}\int_\Omega \rho_0|\boldsymbol{u}|^2 = \int_\Omega \rho_0\,\omega\,\psi$ along the trajectories of the inviscid ($\mu_c = 0$) dynamical system is given by:

$$\frac{d}{dt}\mathcal{H}(t) = \int_{\partial\Omega} \omega\,\psi\,\boldsymbol{n}\cdot\mathbf{grad}^\perp(\psi)\,, \qquad (12)$$

where we can identify, with (10), that $u_n = \boldsymbol{n}\cdot\mathbf{grad}^\perp(\psi)$, and $y_n = e_d = \omega\,\psi = -\psi\,\Delta\psi$.

Now, when $\mu_c > 0$, one can finally write the dissipative dynamical system in the classical pHs form:

$$\begin{pmatrix} \rho_0\partial_t\omega \\ f_c \end{pmatrix} = \begin{bmatrix} J_\omega & -\mathrm{curl}_{2D}\mathbf{grad}^\perp \\ \mathrm{curl}_{2D}\mathbf{grad}^\perp & 0 \end{bmatrix} \begin{pmatrix} e_\omega \\ e_c \end{pmatrix}, \qquad (13)$$

with $\omega = \mathrm{curl}_{2D}\boldsymbol{u}$, $e_\omega = \psi$ and $e_c = \mu_c\omega$, together with the constitutive relation $e_c = \mu_c f_c$.

For the power balance, the computation using (11) gives:

Theorem 5. The evolution of the Hamiltonian along the trajectories of dynamical system (13) with the closure relation is given by:

$$\begin{aligned} \frac{d}{dt}\mathcal{H}(t) = &-\int_\Omega \mu_c\,\omega^2 + \int_{\partial\Omega} \omega\,\psi\,\boldsymbol{n}\cdot\mathbf{grad}^\perp(\psi) \\ &+\mu_c\int_{\partial\Omega} (\psi\,\boldsymbol{n}\cdot\mathbf{grad}\omega - \omega\,\boldsymbol{n}\cdot\mathbf{grad}\psi)\,, \end{aligned} \qquad (14)$$

where we can now identify $u_\tau = \boldsymbol{n}\cdot\mathbf{grad}\psi$, and $y_\tau = -\mu_c\,\omega = \mu_c\,\Delta\psi$.

Remark 6. Note that both controls u_n and u_τ are now available in this formulation. However, another term appears in (14), namely $y_c := \mu_c \boldsymbol{n} \cdot \mathbf{grad}\omega$, the physical meaning of which is not clear so far, but indeed corresponds to the collocated output of the trace of e_ω according to (14). Noticing this is crucial to successfully apply PFEM, see (17) for more details.

Remark 7. A straightforward computation shows that $\mathrm{curl}_{2D}\,\mathbf{grad}^\perp = -\Delta$, minus the 2D scalar Laplacian.

Remark 8. In (13), one can get rid of the realization of dissipation thanks to dissipative ports, and find the dissipative dynamics in the classical form $J - GSG^*$:

$$\rho_0 \partial_t \omega = (J_\omega)\psi - \mu\,\Delta^2\psi\,, \quad \text{with} \quad \psi = \delta_\omega \mathcal{H}\,. \tag{15}$$

Remark 9. Now the 2D incompressible Navier-Stokes equations depend only on 2 scalar fields, in comparison with the former velocity formulation which relied on one vector field and two scalar fields. At the discrete level, this reduces the number of degrees of freedom considerably.

4. STRUCTURE-PRESERVING NUMERICS FOR THE INCOMPRESSIBLE FLUID IN 2D

4.1 Velocity formulation

In this first case where we directly obtain system (9) from (Mora et al., 2020, §. 3), one can of course apply PFEM, following *e.g.* Serhani et al. (2019b). Nevertheless, as already mentionned, this strategy will generate a system with a large number of degrees of freedom.

Furthermore, the total pressure e_d, *i.e.* the Lagrange multiplier of the divergence-free condition $0 = f_d$, is still present in the equation. Since it is only determined up to an additive constant, it is well-known that it has to be taken into account at the discrete level, *e.g.* by fixing its value at one point of the mesh, or by adding an extra Lagrange multiplier to impose a zero-mean pressure in Ω.

The Vorticity-Stream function formulation puts these difficulties aside, for a lower computational price to pay, at least in the 2D setting.

4.2 Vorticity–Stream function formulation

The aim of this work is to take advantage of the 2D setting, in order to reduce as much as possible the number of degrees of freedom needed to discretize the incompressible Navier-Stokes equation in a structure-preserving way . As already said at the end of Section 3.2, some adaptation is required for the boundary controls to remain identical to those of the initial system, but this does not seem to be a major drawback, numerically speaking.

Substituting the constitutive relation $e_c = \mu_c f_c$ in (13) as $\mu_c^{-1}\, e_c = f_c$, the final system reads:

$$\begin{pmatrix} \rho_0\, \partial_t \omega \\ \mu_c^{-1}\, e_c \end{pmatrix} = \begin{bmatrix} J_\omega & -\mathrm{curl}_{2D}\,\mathbf{grad}^\perp \\ \mathrm{curl}_{2D}\,\mathbf{grad}^\perp & 0 \end{bmatrix} \begin{pmatrix} e_\omega \\ e_c \end{pmatrix}. \tag{16}$$

We now apply the Partitioned Finite Element Method. Let us start by the weak formulation of (16). For all sufficiently smooth test functions (v, ϕ), one has:

$$\begin{cases} \displaystyle\int_\Omega \rho_0\, \partial_t\omega\; v\; \mathrm{d}x = \int_\Omega J_\omega e_\omega\; v\; \mathrm{d}x \\ \qquad\qquad\qquad\qquad - \displaystyle\int_\Omega \mathrm{curl}_{2D}\,\mathbf{grad}^\perp(e_c)\; v\; \mathrm{d}x, \\ \displaystyle\int_\Omega \mu_c^{-1}\; e_c\; \phi\; \mathrm{d}x = \int_\Omega \mathrm{curl}_{2D}\,\mathbf{grad}^\perp(e_\omega)\; \phi\; \mathrm{d}x. \end{cases}$$

Applying Green's formula on the integral involving J_ω, the integration by parts (11) on integrals involving $\mathrm{curl}_{2D}\,\mathbf{grad}^\perp$ and noting that $\mathcal{R}\,\mathbf{grad}^\perp = -\mathbf{grad}$, gives:

$$\begin{cases} \displaystyle\int_\Omega \rho_0\,\partial_t\omega\, v\,\mathrm{d}x = -\int_\Omega \omega\,\mathbf{grad}^\perp(e_\omega)\cdot\mathbf{grad}(v)\;\mathrm{d}x \\ \qquad\qquad + \displaystyle\int_{\partial\Omega} \omega\, \underbrace{\mathbf{grad}^\perp(e_\omega)\cdot\boldsymbol{n}}_{\boldsymbol{u}\cdot\boldsymbol{n}=:u_n} v\,\mathrm{d}s \\ \qquad\qquad - \displaystyle\int_\Omega \mathbf{grad}^\perp(e_c)\cdot\mathbf{grad}^\perp(v)\;\mathrm{d}x \\ \qquad\qquad + \displaystyle\int_{\partial\Omega} \underbrace{(\mathbf{grad}(e_c))\cdot\boldsymbol{n}}_{y_c}\, v\,\mathrm{d}s, \\ \displaystyle\int_\Omega \mu_c^{-1}\, e_c\,\phi\,\mathrm{d}x = \int_\Omega \mathbf{grad}^\perp(e_\omega)\cdot\mathbf{grad}^\perp(\phi)\;\mathrm{d}x \\ \qquad\qquad - \displaystyle\int_{\partial\Omega} \underbrace{(\mathbf{grad}(e_\omega))\cdot\boldsymbol{n}}_{\boldsymbol{u}\wedge\boldsymbol{n}=:u_\tau}\,\phi\,\mathrm{d}s. \end{cases} \tag{17}$$

Normal and tangential boundary controls for the velocity (u_n and u_τ respectively) are now available in the weak formulation.

Let $(v^j)_{j=1,\cdots,N_\omega}$ (resp. $(\phi^\ell)_{\ell=1,\cdots,N_c}$) be a finite element basis of approximation for ω and ψ (resp. e_c). We denote:

$$\omega^d(t,x) = \sum_{j=1}^{N_\omega} \omega^j(t) v^j(x), \qquad \psi^d(t,x) = \sum_{j=1}^{N_\omega} \psi^j(t) v^j(x),$$
$$e_c^d(t,x) = \sum_{\ell=1}^{N_c} e_c^\ell(t)\phi^\ell(x),$$

the approximations of ω, ψ and e_c respectively. Let also $(\xi^n)_{n=1,\cdots,N_\partial}$ be a finite element basis for the approximation of u_n, y_c and u_τ on the boundary $\partial\Omega$ (chosen identical, for sake of simplicity).

The discrete weak formulation is then given by: taking $v = v^i$ for all $i \in \{1,\cdots,N_\omega\}$ and $\phi = \phi^k$ for all $k \in \{1,\cdots,N_c\}$ as test functions:

$$\begin{bmatrix} M_\omega & 0 \\ 0 & M_c \end{bmatrix} \begin{pmatrix} \dot{\underline{\omega}} \\ \dot{\underline{e_c}} \end{pmatrix} = \begin{bmatrix} J_\omega(\omega^d) & -D \\ D^\top & 0 \end{bmatrix} \begin{pmatrix} \underline{e_\omega} \\ \underline{e_c} \end{pmatrix} + \begin{bmatrix} B_n & B_c & 0 \\ 0 & 0 & B_\tau \end{bmatrix} \begin{pmatrix} \underline{u_n} \\ \underline{y_c} \\ \underline{u_\tau} \end{pmatrix},$$

where $\underline{\square}$ is the collection of the time-dependent coefficients of the approximation $\square^d$ in the associated finite element basis, a dot denotes the time derivative, and:

$$(M_\omega)_{i,j} = \int_\Omega \rho_0 v^j v^i\;\mathrm{d}x, \quad (M_c)_{k,\ell} = \int_\Omega \mu_c^{-1}\phi^\ell\phi^k\;\mathrm{d}x,$$
$$(J_\omega(\omega^d))_{i,j} = \int_\Omega \omega^d\,\mathbf{grad}^\perp\left(v^j\right)\cdot\mathbf{grad}\left(v^i\right)\;\mathrm{d}x,$$
$$(D)_{i,\ell} = \int_\Omega \mathbf{grad}^\perp(\phi^\ell)\cdot\mathbf{grad}^\perp(v^i)\;\mathrm{d}x,$$
$$(B_n)_{i,n} = \int_{\partial\Omega} \omega^d\,\xi^n\,v^i\;\mathrm{d}x, \quad (B_c)_{i,n} = \int_{\partial\Omega} \xi^n\,v^i\;\mathrm{d}x,$$
$$(B_\tau)_{k,n} = -\int_{\partial\Omega} \xi^n\,\phi^k\;\mathrm{d}x.$$

Note that $D \in \mathbb{R}^{N_\omega \times N_c}$ is not square in general (as $B_n, B_c \in \mathbb{R}^{N_\omega \times N_\partial}$ and $B_\tau \in \mathbb{R}^{N_c \times N_\partial}$).

Remark 10. Interestingly, integration by parts has been here performed on *both* lines, while PFEM usually relies on one integration by parts on the appropriate line (depending on the considered causality).

About ω and e_ω: unfortunately, a huge difficulty appears in the previous system, since the constitutive relation between ω and e_ω is differential, namely $-\Delta e_\omega = \omega$ and not algebraix. Several strategies could be considered to tackle this numerically.

Let us first point out why adding the constitutive relation as an extra constraint would not solve the problem: indeed, the extra constraint which would be necessary, $-\Delta e_\omega = \omega$, is already present in (16): it is exactly the second line of the port-Hamiltonian structure, hence it would not help to solve the problem.

Now, a second approach seems to be the most promising in our opinion: the idea relies on a general comment on PFEM: the *co-energy formulation*, *i.e.* taking all constitutive relations into account at the continuous level, such that the system only writes in term of co-energy variables, and still allows dealing with sparse matrices only. Indeed, with the *energy formulation*, matrix inversions would be necessary to reduce the finite-dimensional system. An example of that occurs if one first discretizes $e_c = \mu_c f_c$, before its substitution in (16). If the same idea of substitution before discretisation is applied with the differential constitutive relation, one obtains:

$$\begin{bmatrix} \widetilde{M}_\omega & 0 \\ 0 & M_c \end{bmatrix} \begin{pmatrix} \dot{\underline{e_\omega}} \\ \dot{\underline{e_c}} \end{pmatrix} = \begin{bmatrix} J_\omega(\omega^d) & -D \\ D^\top & 0 \end{bmatrix} \begin{pmatrix} \underline{e_\omega} \\ \underline{e_c} \end{pmatrix} + \begin{bmatrix} B_n & B_c & 0 & \widetilde{B}_\tau \\ 0 & 0 & B_\tau & 0 \end{bmatrix} \begin{pmatrix} \underline{u_n} \\ \underline{y_c} \\ \underline{u_\tau} \\ \dot{\underline{u_\tau}} \end{pmatrix},$$

where $(\widetilde{M}_\omega)_{i,j} := \int_\Omega \rho_0 \, \mathbf{grad}(v^j) \cdot \mathbf{grad}(v^i) \, \mathrm{d}x$, and $(\widetilde{B}_\tau)_{i,n} := \int_{\partial\Omega} \rho_0 \xi^n \, v^i \, \mathrm{d}x$. Note that $\widetilde{M}_\omega$ is symmetric positive, and even positive-definite as soon as stream functions are well-defined, for instance in simply connected domains. According to Brugnoli et al. (2020), $\underline{y_c}$ is the Lagrange multiplier of the boundary control $\underline{u_c}$ of $e_\omega = \psi$, which clearly depends (up to a constant) on $\underline{u_n}$ and $\underline{u_\tau}$.

The problem of relating ω and e_ω being solved, this strategy requires a more thorough study to conclude that it is promising on the computational side. Indeed, the system can prove difficult to solve, since the mass matrix $\widetilde{M}_\omega$ is a stiffness-like matrix. Also, one can expect that some compatibility conditions between the finite element bases are mandatory.

About $J_\omega(\omega^d)$: At first sight, the ω^d-dependent matrix $J_\omega(\omega^d)$ will obviously generate a difficulty, especially for the time stepping of the finite-dimensional pHs. For instance, it seems difficult to make use of an implicit scheme as such: the classical approach is to update the matrix using the approximation ω^d at the previous time step (not necessarily with a full re-assembling, but still). We propose here another novel approach, based on tensor algebra.

Let us come back to:

$$(J_\omega(\omega^d))_{i,j} = \int_\Omega \omega^d \, \mathbf{grad}^\perp\left(v^j\right) \cdot \mathbf{grad}\left(v^i\right) \, \mathrm{d}x,$$

and develop the approximation ω^d:

$$(J_\omega(\omega^d))_{i,j} = \sum_{\kappa=1}^{N_\omega} \omega^\kappa(t) \int_\Omega v^\kappa \, \mathbf{grad}^\perp\left(v^j\right) \cdot \mathbf{grad}\left(v^i\right) \, \mathrm{d}x,$$

This rewrites: $(J_\omega(\omega^d(t)))_{i,j} = \left(\overline{J}_\omega\right)_{i,j,\kappa} \omega^\kappa(t)$, where $\overline{J}_\omega \in \mathbb{R}^{N_\omega^3}$ is a *constant* third-order tensor.

The size of this tensor, N_ω^3 coefficients, seems to be a major drawback in terms of memory for efficient computations. However, the sparsity property of the matrices generated by the finite element method remains true, and $\overline{J}_\omega$ is indeed (very!) sparse, allowing for optimized computations. Of course, the same strategy can be apply to B_n.

Remark 11. To the best of our knowledge, the idea consisting of considering a polynomial non-linearity as a multilinear application when dealing with finite elements is new.

This also reveals major advantages when dealing with non-linearity in the constitutive relations, but this is not the scope of this work.

5. CONCLUSION AND PERSPECTIVES

The formulation of the incompressible Navier-Stokes equations with collocated control and observation has been recalled in two forms: either with respect to the velocity, or with respect to the vorticity. In the 2D case, the pHs involves only 2 scalar fields, which is an advantage for simulation. However the interconnection matrix depends on vorticity, which is a drawback. A novel application of the Partitioned Finite Element Method to this latter case has been proposed, implementation considerations have been listed, and first convincing simulations results have already been obtained.

Appendix A. USEFUL IDENTITIES

Computing the convective term in 3D makes use of:

$$(\boldsymbol{u} \cdot \mathbf{grad})\boldsymbol{u} = \mathbf{grad}(\frac{1}{2}|\boldsymbol{u}|^2) + \mathbf{curl}(\boldsymbol{u}) \wedge \boldsymbol{u}\,. \tag{A.1}$$

Hence, setting $\boldsymbol{\omega} := \mathbf{curl}(\boldsymbol{u})$ the vorticity vector, we can define

$$G(\boldsymbol{\omega}) := \boldsymbol{\omega}\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_2 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

In dimension 2, $\boldsymbol{\omega} = \omega_3 \boldsymbol{k}$, matrix $G(\boldsymbol{\omega})$ simplifies into

$$G(\omega) := \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

with $\omega := \mathrm{curl}_{2D}(\boldsymbol{u})$, and identity (A.1) simplifies into

$$(\boldsymbol{u} \cdot \mathbf{grad})\boldsymbol{u} = \mathbf{grad}(\frac{1}{2}|\boldsymbol{u}|^2) - \omega\,\mathcal{R}\,\boldsymbol{u}\,, \tag{A.2}$$

where $\mathcal{R}$ is the rotation of angle $-\frac{\pi}{2}$: $\mathcal{R}(u_1, u_2) = (u_2, -u_1)$.

In dimension 3, another useful identity comes from two expressions to be found for $\mathbf{C}\cdot(\mathbf{E}\wedge\mathbf{H}) = \mathbf{H}\cdot(\mathbf{C}\wedge\mathbf{E}) = -\mathbf{E}\cdot(\mathbf{C}\wedge\mathbf{H})$, and applied with the derivation operator ∇, which gives in turn:

$$\mathrm{div}(\mathbf{E} \wedge \mathbf{H}) = \mathbf{H} \cdot \mathrm{curl}\,\mathbf{E} - \mathbf{E} \cdot \mathrm{curl}\,\mathbf{H}\,. \tag{A.3}$$

An immediate consequence of this identity is its integral version, involving Stokes formula:

$$\int_\Omega (\mathbf{E} \cdot \mathrm{curl}\,\mathbf{H} - \mathbf{H} \cdot \mathrm{curl}\,\mathbf{E}) = -\int_{\partial\Omega} \boldsymbol{\Pi} \cdot \mathbf{n}\,, \tag{A.4}$$

introducing the *Poynting vector* $\boldsymbol{\Pi} := \gamma(\mathbf{E}\wedge\mathbf{H})$ defined on the boundary $\partial\Omega$.

Appendix B. FACTORIZING OUT THE OPPOSITE OF THE VECTORIAL LAPLACIAN

For the vectorial Laplacian, both in dimension 2 and 3, the following factorizations prove useful.

Proposition 12. In dimension 2, the vectorial Laplacian can be factorized out as follows:

$$-\boldsymbol{\Delta}_{2D} := \mathbf{grad}^{\perp}\mathrm{curl}_{2D} - \mathbf{grad}\,\mathrm{div} \tag{B.1}$$
$$= C_c^* C_c + C_d^* C_d = \boldsymbol{C}^*\boldsymbol{C} \tag{B.2}$$

Proof. The proof is straightforward, and uses the fact that the formal adjoint of $\mathrm{curl}_{2D} := [-\partial_y \ \partial_x]$ is $\mathrm{curl}_{2D}^* = \begin{bmatrix} \partial_y \\ -\partial_x \end{bmatrix} := \mathbf{grad}^{\perp}$. □

Proposition 13. In dimension 3, the vectorial Laplacian can be factorized out as follows:

$$-\boldsymbol{\Delta} := \mathbf{curl}\,\mathbf{curl} - \mathbf{grad}\,\mathrm{div} \tag{B.3}$$
$$= C_c^* C_c + C_d^* C_d = \boldsymbol{C}^*\boldsymbol{C} \tag{B.4}$$

Proof. The proof is straightforward, and uses the fact that the formal adjoint of **curl** is **curl**. □

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