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IFAC PapersOnLine 53-2 (2020) 11503–11508

Modelling the 1D piston problem as interconnected port-Hamiltonian systems ⋆

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Abstract: In this study, the modelling of the boundary-controlled 1D piston problem as the interconnection of simpler port-Hamiltonian systems (pHs) is addressed. More precisely, two viscous compressible fluids are separated by a moving rigid body on a bounded domain $(0, L)$. Thermodynamics is taken into account, leading to two pHs for each physical domain: one associated to the kinetic energy and the other one to the internal energy. No chemical reaction is being considered in the system. A control by mass injection/rejection and heating is then applied at the left end of the first fluid.

Keywords: Piston problem; port-Hamiltonian system; Dirac structure; free boundary; non-linear PDE; fluid-structure-thermal interactions; control and observation.

1. INTRODUCTION

Port-Hamiltonian systems (pHs), introduced a few decades ago, see *e.g.* van der Schaft and Maschke (2002); Duindam et al. (2009); van der Schaft and Jeltsema (2014), are a powerful tool to represent complex physical systems, based on exchanges of energy between their components. Very different multi-physics applications can be described through it, *e.g.* plasmas in tokamaks Vu et al. (2016), or fluid structure interaction Cardoso-Ribeiro et al. (2017). Furthermore, pHs are strongly related to Dirac structure, allowing for power-preserving interconnections, which are easy to state.

In this work, we propose to model the boundary-controlled 1D piston problem, composed of three physical subsystems interacting between them on a bounded domain $(0, L)$ (see Figure 1). The first difficulty is that the domain of each sub-system is a free boundary problem, already tackled *e.g.* in Diagne and Maschke (2013). At the end of the modelling process, comportemental laws will be added to define a rigid body between two viscous fluids, with thermal diffusion given by Fourier's law. This physical system has been widely studied in recent years, with various points of view, see for instance Ding and Li (2014); Lequeurre and Tucsnak (2015); Ndanou et al. (2017); Maity et al. (2017); van der Schaft and Maschke (2018).

The proposed strategy consists in taking full advantage of the pHs formalism: the total energy of the piston system is decomposed into three kinetic energies and three internal energies (*i.e.* two kind of energies for each physical domain), allowing exchanges between them. Each of these energies is modelled through an underlying Dirac structure, defining flows, efforts and an extended structure operator. The resulting interconnected system will remain a port-Hamiltonian system, as a composition of Dirac structures (see *e.g.* Kurula et al. (2010)). Finally, a control by mass injection/rejection and heating is applied at the left end.

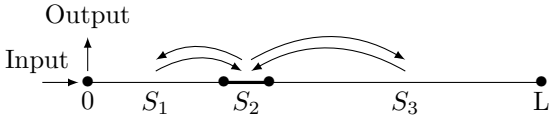

Fig. 1. The 1D model of a piston. Sub-systems S_1 and S_3 (viscous fluids) act on S_2 (rigid body) while subsystem S_2 acts on S_1 and S_3. S_1 is controlled and observed at the left end. Each arrow represents an interaction.

In this work, we start with the only assumption that each subsystem is a continuous medium, satisfying mass, linear momentum and energy conservations (as soon as they are closed); moreover, no chemical reaction is to be found in any of them. The addition of physical hypotheses such as phenomenological laws (*e.g.* definition of the stress tensor for S_1 and S_3, assumption for S_2 to be a rigid body) is then postponed as far as possible in the modelling process.

1.1 Hamiltonian and underlying Dirac structure

There are several ways to define a port-Hamiltonian system: it can be done explicitly or implicitly. In this work, we focus on the *Hamiltonian* $\mathcal{H}$ as the primary object to model a system.

The Hamiltonian depends on *energy variables* $\overrightarrow{\alpha}$ and is often taken to be the energy of the system. The variational derivative $\delta_{\overrightarrow{\alpha}}\mathcal{H}$ of $\mathcal{H}$ w.r.t. $\overrightarrow{\alpha}$ defines the *co-energy variables* $\overrightarrow{e}_{\overrightarrow{\alpha}}$. Using evolution equations satisfied by the energy and co-energy variables, one can derive a *power balance*, *i.e.* an equality expressing the variation in time of the Hamiltonian (and in particular how to make it vary

⋆ This work is supported by the project n° ANR-16-CE92-0028, entitled *Interconnected Infinite-Dimensional systems for Heterogeneous Media*, INFIDHEM, financed by the French National Research Agency (ANR) and the Deutsche Forschungsgemeinschaft (DFG). Further information is available at `https://websites.isae-supaero.fr/infidhem/the-project`.

10.1016/j.ifacol.2020.12.601

for control purposes). Theoretically, it is strongly related to an algebraic structure: an *underlying Dirac structure*.

In this work, we identify for *both kinetic and internal energies* a Hilbert space $\mathcal{X}$ (the *state space*) and a Hilbert space $\mathcal{E}$ (the *effort space*) which is dense in $\mathcal{X}$. Then defining $\mathcal{F}$ (the *flow space*) as the topological dual of $\mathcal{E}$ w.r.t. the pivot space $\mathcal{X}$, we rewrite the evolution equations satisfied by $\overrightarrow{\boldsymbol{\alpha}}$ and $\overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}$ in an equality $\overrightarrow{\boldsymbol{f}} = \mathcal{J}\overrightarrow{\boldsymbol{e}}$, where $\mathcal{J} \in \mathcal{L}(\mathcal{E}_1, \mathcal{F})$ is a closed and densely defined operator such that $\left\langle \mathcal{J}\overrightarrow{\boldsymbol{e}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}} = -\left\langle \mathcal{J}\widetilde{\overrightarrow{\boldsymbol{e}}}, \overrightarrow{\boldsymbol{e}} \right\rangle_{\mathcal{F},\mathcal{E}}$ for all $\overrightarrow{\boldsymbol{e}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \in \mathcal{E}_1$.

Theorem 1. Let $\mathcal{J}$ be as above, then $\mathcal{D} := \mathrm{Graph}(\mathcal{J}) := \left\{ \left(\overrightarrow{\boldsymbol{f}}, \overrightarrow{\boldsymbol{e}} \right) \in \mathcal{B} \mid \overrightarrow{\boldsymbol{f}} = \mathcal{J}\overrightarrow{\boldsymbol{e}} \right\}$ is a Dirac structure on the Bond space $\mathcal{B} := \mathcal{F} \times \mathcal{E}$, endowed with the symmetrized bilinear product

$$\left[\left(\overrightarrow{\boldsymbol{f}}_1, \overrightarrow{\boldsymbol{e}}_1 \right), \left(\overrightarrow{\boldsymbol{f}}_2, \overrightarrow{\boldsymbol{e}}_2 \right) \right]_{\mathcal{B}} := \left\langle \overrightarrow{\boldsymbol{f}}_1, \overrightarrow{\boldsymbol{e}}_2 \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \overrightarrow{\boldsymbol{f}}_2, \overrightarrow{\boldsymbol{e}}_1 \right\rangle_{\mathcal{F},\mathcal{E}}. \quad (1)$$

In other words, $\mathcal{D}$ is maximal isotropic: $\mathcal{D}^{[\perp]} = \mathcal{D}$, where the orthogonal companion $\mathcal{D}^{[\perp]}$ is taken w.r.t. (1). The operator $\mathcal{J}$ is called the *extended structure operator* of $\mathcal{D}$. The proof is given in Appendix A.

An element $\left(\overrightarrow{\boldsymbol{f}}_s, \overrightarrow{\boldsymbol{e}}_s \right) \in \mathcal{D}$ will be called a *state solution* if the first component of $\overrightarrow{\boldsymbol{f}}_s$ is the time derivative of the energy variables, and the first component of $\overrightarrow{\boldsymbol{e}}_s$ is the co-energy variables. In other words, if one has $\overrightarrow{\boldsymbol{f}}_s = \left(\partial_t \overrightarrow{\boldsymbol{\alpha}}, \cdots \right)^\top$ and $\overrightarrow{\boldsymbol{e}}_s := \left(\overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}, \cdots \right)^\top$.

Such a Dirac structure will be called an *underlying Dirac structure* for the Hamiltonian $\mathcal{H}$ if furthermore the power balance can be recovered from the identity $\left\langle \overrightarrow{\boldsymbol{f}}_s, \overrightarrow{\boldsymbol{e}}_s \right\rangle_{\mathcal{F},\mathcal{E}} = 0$ (a concrete example of this definition is given in the proof of Theorem 2).

Remark 1. It has to be noted that the underlying Dirac structure is far from unique in general, since it heavily depends on the choice of the energy variables. Furthermore, if a state solution exists, it is not necessarily unique: *closure equations* (or *constitutive relations*) relating each component of flows and efforts are needed to ensure uniqueness.

Remark 2. Another important point is that state solutions do not have to exist to define an underlying Dirac structure. It is a formal computation allowing to characterize $\mathcal{D}$ in a common way. Indeed, since we deal with boundary control systems, several difficulties arise and the solution is in general a *weak solution*, belonging to the subspace $\mathcal{E}_1$ (*i.e.* the *solution space*), see *e.g.* (Tucsnak and Weiss, 2009, Chapter 10) for a thorough introduction to boundary control systems.

1.2 Physical principles and modelling

Consider a continuous medium, occupying a 1D moving domain $(a(t), b(t)) \in \mathbb{R}$ for all $t \geq 0$, and characterized by

- its linear mass density $\rho(t,x)$;
- its particle velocity $v(t,x)$;
- its stress $\sigma(t,x)$;
- its heat flux $J_q(t,x)$.

This continuous medium is supposed to satisfy the classical laws of conservation in physics, namely: conservation of mass, conservation of linear momentum $p(t,x) := \rho(t,x)v(t,x)$ and conservation of internal energy $\overline{u}(t,x) := \rho(t,x)u(t,x)$, with u the internal energy density. Following Boyer and Fabrie (2013), these assumptions read for all $t \geq 0$ and all $x \in (a(t), b(t))$

$$\partial_t \rho(t,x) = -\partial_x \left(p(t,x) \right), \quad (2)$$

$$\partial_t p(t,x) = -\partial_x \left(p(t,x)v(t,x) \right) + \partial_x \sigma(t,x), \quad (3)$$

$$\partial_t \left(\overline{u}(t,x) \right) = -\partial_x \left(\overline{u}(t,x)v(t,x) \right) + \sigma(t,x)\,\partial_x v(t,x) - \partial_x J_q(t,x). \quad (4)$$

Finally, since chemical reactions are not included in this modelling, Gibbs' formula reads

$$\mathrm{d}\overline{u} = T\ \mathrm{d}\overline{s} - \pi\ \mathrm{d}V, \quad (5)$$

where d is the infinitesimal increment, T is the local temperature, $\overline{s} = \rho s$ is the local entropy, s the entropy density, π the local pressure and V the volume (namely the length $b(t) - a(t)$ in the 1D case).

From the evolution equation of the total energy given by the sum of the kinetic and internal energies, the previous evolution equations and Gibbs' formula (5) lead to the following evolution equation for the entropy

$$\partial_t \left(\overline{s}(t,x) \right) = -\partial_x \left(\overline{s}(t,x)v(t,x) \right) - \partial_x \left(J_s(t,x) \right) + \Sigma(t,x), \quad (6)$$

where $J_s := \frac{1}{T} J_q$ is the *entropy flux* and, with $\mathrm{d}_t := \frac{\mathrm{d}}{\mathrm{d}t}$,

$$\Sigma := \frac{1}{T} \left(\sigma \partial_x v - J_s \partial_x T + \pi\ \mathrm{d}_t V \right), \quad (7)$$

is the *irreversible entropy production*, which is positive thanks to Clausius–Duhem inequality: this is the second law of thermodynamics.

1.3 Organization of the paper

In Section 2, the kinetic energy of a continuous medium is depicted as a pHs. First, the Hamiltonian and its power balance are given, then interconnection ports are identified, and the underlying Dirac structure is constructed. In Section 3, the same approach is applied to the internal energy. Section 4 presents the main result of this work: the piston problem as interconnection of elementary pHs. At the end, conclusions and further works are given in Section 5.

The following and easy-to-prove formula will be useful in the sequel: let $F(t) := \int_{a(t)}^{b(t)} f(t,x)\ \mathrm{d}x$, with smooth enough functions. Then

$$\mathrm{d}_t F(t) = \mathrm{d}_t b(t) f(t, b(t)) - \mathrm{d}_t a(t) f(t, a(t)) + \int_{a(t)}^{b(t)} \partial_t f(t,x)\ \mathrm{d}x. \quad (8)$$

This is the celebrated *transport theorem*, written in 1D.

2. KINETIC ENERGY

This section aims at depicting the kinetic energy as a pHs.

2.1 Hamiltonian and power balance

The kinetic energy is given by

$$\mathcal{H}_k(p, \rho) := \int_{a(t)}^{b(t)} \frac{p^2(t,x)}{2\rho(t,x)}\ \mathrm{d}x, \quad \forall t \geq 0. \quad (9)$$

It is chosen as the Hamiltonian functional, with the linear momentum p and mass density ρ as energy variables. The co-energy variables are by definition the variational derivative of $\mathcal{H}_k$ w.r.t. the energy variables: *i.e.* $\delta_p\mathcal{H}(p,\rho) = \frac{p}{\rho} = v$, the velocity in the medium, and $\delta_\rho\mathcal{H}_k(p,\rho) = -\frac{p^2}{2\rho^2} = -\frac{v^2}{2}$, the flux appearing in the so-called Burgers' equation (which can also be seen as *minus the kinetic energy density*).
One easily computes, thanks to (8), (2) and (3), the following power balance for all $t \geq 0$

$$\begin{aligned}\mathrm{d}_t\mathcal{H}_k(p,\rho) = \rho(t,b(t))\left(v(t,b(t)) - \mathrm{d}_t b(t)\right)\left(-\frac{v^2(t,b(t))}{2}\right)\\ - \rho(t,a(t))\left(v(t,a(t)) - \mathrm{d}_t a(t)\right)\left(-\frac{v^2(t,a(t))}{2}\right)\\ + v(t,b(t))\sigma(t,b(t)) - v(t,a(t))\sigma(t,a(t))\\ - \int_{a(t)}^{b(t)} \partial_x v(t,x)\sigma(t,x)\,\mathrm{d}x. \quad (10)\end{aligned}$$

This power balance expresses the various ways to make the kinetic energy vary; moreover it indicates the relevant physical connection ports which are available for further energy exchanges.

2.2 Interconnection ports

The ports allowing energy exchanges are defined. They are of two kinds. The first one concerns ports that are not accessible to an external operator, and model for instance the transformation of the kinetic energy into internal energy. The other type of interconnection ports are those allowing the physical system to exchange energy with the environment. In infinite dimension, the former are often distributed inside the domain, while the latter are often defined at the boundary.

Looking at the power balance (10) for the kinetic energy $\mathcal{H}_k$, it can be seen that the L^2-product of the flow $\partial_x v$ by the effort σ corresponds to a distributed variation of energy. If we consider the particular case of a Newtonian fluid, this will lead to a dissipation (related to the kinematic viscosity), with a loss of kinetic energy toward the internal energy. We will call these flow and effort the *dissipative ports*. Note that the thermal dissipation due to shocks are also taken into account in this term.

Physically speaking, it seems more relevant to control the pressure and the rate of flow at the boundary: we want to control the dynamical equations (2) and (3) by boundary conditions. As a consequence, the particle velocity at the boundary is not necessarily the velocity of the boundary, for instance if mass is being injected or rejected. This induces three possible boundary controls: the pressure, the rate of flow and the velocity of the boundary itself. Let us denote the boundary trace linear operators from $H^1(a,b)$ onto $\mathbb{R}^2$: $\gamma_\perp(f) := \begin{pmatrix} -f(a) \\ f(b)\end{pmatrix}$ and $\gamma_0(f) := \begin{pmatrix} f(a) \\ f(b)\end{pmatrix}$, as well as $\Gamma(t) := \begin{pmatrix} -\rho(t,a(t))\,\mathrm{d}_t a(t) \\ \rho(t,b(t))\,\mathrm{d}_t b(t)\end{pmatrix}$. The boundary controls and observations are defined by

$$\begin{aligned} u_\sigma &:= \gamma_\perp(\sigma), \quad & y_\sigma &:= \gamma_0(v),\\ u_r &:= \gamma_\perp(\rho v) - \Gamma, \quad & y_r &:= \gamma_0\left(-\frac{v^2}{2}\right). \end{aligned} \quad (11)$$

With these notations at hand, the energy balance (10) now reads

$$\mathrm{d}_t\mathcal{H}_k(p,\rho) = \langle u_r, y_r\rangle_{\mathbb{R}^2} + \langle u_\sigma, y_\sigma\rangle_{\mathbb{R}^2} - \langle \partial_x v, \sigma\rangle_{L^2}. \quad (12)$$

2.3 An underlying Dirac structure

Following the previous section, flows and efforts obtained from the evolution equations can be extended using the control and observation ports allowing for exchange of energy. Let us define

$$\overrightarrow{\boldsymbol{f}}_k := \left(\partial_t p,\ \partial_t\rho,\ -\partial_x v,\ \partial_x v,\ \partial_x\left(\frac{v^2}{2}\right),\ -y_\sigma,\ -y_r\right)^\top,$$
$$\overrightarrow{\boldsymbol{e}}_k := \left(v,\ -\frac{v^2}{2},\ pv,\ \sigma,\ \rho v,\ u_\sigma,\ u_r\right)^\top.$$

Formally, the structure operator is then given by

$$\mathcal{J}_k := \begin{pmatrix} 0 & 0 & -\partial_x & \partial_x & 0 & \gamma_0^* & 0\\ 0 & 0 & 0 & 0 & -\partial_x & 0 & \gamma_0^*\\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0\\ \partial_x & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\partial_x & 0 & 0 & 0 & 0 & 0\\ -\gamma_0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\gamma_0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (13)$$

where $\gamma_0^* \in \mathcal{L}\left(\mathbb{R}^2, H^{-1}(a,b)\right)$ is the adjoint of γ_0 (strongly related to the so-called *Dirichlet map*, see *e.g.* Tucsnak and Weiss (2009)). Here $H^{-1}(a,b)$ denotes the dual of $H_0^1(a,b)$ w.r.t. the pivot space $L^2(a,b)$.

However, (13) is not rigorous. The use of *boundary control systems theory* (see *e.g.* (Tucsnak and Weiss, 2009, Chapter 10)) and *system nodes* (see *e.g.* Malinen et al. (2006); Le Gorrec et al. (2004)) is necessary to obtain a properly defined structure operator. Nevertheless, the formal definition (13) is sufficient for our purpose.

In order to properly determine the effort space $\mathcal{E}_k$ and its important dense subspace $\mathcal{E}_1^k$, namely the domain of the system node given by (13), we follow (Tucsnak and Weiss, 2009, Chapter 10) and Malinen et al. (2006). Then we define $\mathcal{E}^k := \left(L^2(a,b)\right)^5 \times \left(\mathbb{R}^2\right)^2$, $\mathcal{E}_1^k := \left(H^1(a,b)\right)^5 \times \left(\mathbb{R}^2\right)^2$, and the flow space $\mathcal{F}^k := \left(\mathcal{E}^k\right)' = \left(L^2(a,b)\right)^5 \times \left(\mathbb{R}^2\right)^2$.

Remark 3. Here the state space $\mathcal{X}^k$ defining the pivot space to determined $\mathcal{F}^k$ is directly given as $\mathcal{E}^k$.

Theorem 2. Let the Bond space $\mathcal{B}^k := \mathcal{F}^k \times \mathcal{E}^k$ endowed with the symmetrized product (1), then $\mathcal{D}^k := \mathrm{Graph}\left(\mathcal{J}_k\right)$ is an underlying Dirac structure for (9).

Remark 4. Keep in mind that, a and b being time-dependent, every object in the above theorem might be time-dependent.

Proof. The proof needs the rigorous definition of the structure operator $\mathcal{J}_k$ (for each fixed $t \geq 0$). In particular, the feedthrough operator of the system has to be properly defined (see (Malinen et al., 2006, p. 70)) to prove symmetry, *i.e.* that $\left\langle \mathcal{J}_k \overrightarrow{\boldsymbol{e}}, \widetilde{\overrightarrow{\boldsymbol{e}}}\right\rangle_{\mathcal{F}^k,\mathcal{E}^k} = -\left\langle \mathcal{J}_k \widetilde{\overrightarrow{\boldsymbol{e}}}, \overrightarrow{\boldsymbol{e}}\right\rangle_{\mathcal{F}^k,\mathcal{E}^k}$ for all $\overrightarrow{\boldsymbol{e}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \in \mathcal{E}_1^k$ thanks to the abstract integration by parts formula (Tucsnak and Weiss, 2009, Equation (10.1.7)).

From Theorem 1, $\mathcal{D}^k$ is a Dirac structure on $\mathcal{B}^k$.

To conclude that this Dirac structure is an underlying Dirac structure for the Hamiltonian (9), it remains to show that if $\left(\overrightarrow{\boldsymbol{f}}_k, \overrightarrow{\boldsymbol{e}}_k\right) \in \mathcal{D}^k$ (*i.e.* if this is a state solution), then $\left\langle \overrightarrow{\boldsymbol{f}}_k, \overrightarrow{\boldsymbol{e}}_k \right\rangle_{\mathcal{F}^k,\mathcal{E}^k} = 0$ allows to recover (12). It is straitghforward using (Tucsnak and Weiss, 2009, Equation (10.1.7)).

3. INTERNAL ENERGY

We now aim at modelling the internal energy of the physical domain as a pHs. This model generalizes one of those proposed in Serhani et al. (2019a), where the domain was a rigid body.

3.1 Hamiltonian and power balance

Let us define the Hamiltonian as the internal energy

$$\mathcal{H}_u(\overline{s},\rho) := \int_{a(t)}^{b(t)} \overline{u}\,(t,x)\ \mathrm{d}x, \quad \forall t \geq 0. \tag{14}$$

The energy variables are chosen to be the local entropy $\overline{s}$ and the mass density ρ. Let $f := u - Ts$ be the *Helmholtz free energy density*. Hence $\mathcal{H}_u(\overline{s},\rho) = \int_{a(t)}^{b(t)} \rho(t,x) f(t,x) + \rho(t,x)s(t,x)T(t,x)\ \mathrm{d}x$. The co-energy variables are then given by $\delta_{\overline{s}}\mathcal{H}_u = T$ and $\delta_\rho \mathcal{H}_u = f + Ts = u$.

From (6) it is straightforward to compute for all $t \geq 0$

$$\begin{aligned}\mathrm{d}_t\mathcal{H}_u(\overline{s},\rho) &= \rho(t,b(t))\,(\ \mathrm{d}_t b(t) - v(t,b(t))\ )\,u(t,b(t)) \\ &\quad -\rho(t,a(t))\,(\ \mathrm{d}_t a(t) - v(t,a(t))\ )\,u(t,a(t)) \\ &\quad -J_q(t,b(t)) + J_q(t,a(t)) \\ &\quad + \int_{a(t)}^{b(t)} \partial_x v(t,x)\sigma(t,x)\ \mathrm{d}x.\end{aligned} \tag{15}$$

As in the case of the kinetic energy, four power products appear, which give rise to four ways to make the internal energy vary.

3.2 Interconnection ports

By analyzing (15), it is seen that the integral term is nothing but minus the integral term in (10). It corresponds to the creation of internal energy coming from the transformation of the kinetic energy. This product appears in the irreversible entropy production, and hence does not prove necessary to write in the system under the form $\overrightarrow{\boldsymbol{f}} = \mathcal{J}\overrightarrow{\boldsymbol{e}}$, *i.e.* in an underlying Dirac structure. It will be taken into account in the constitutive relation given by the definition of the irreversible entropy production (7). Consequently, we do not have to define *accretive ports* to cancel the dissipative ports of the kinetic energy system.

As for the kinetic energy, the rate of flow as well as the velocity of the boundary are available for external interactions. The third boundary ports are those involving the heat flux.

Physically speaking, it seems natural to control the temperature at the boundary (which is available through the definition of J_s): we want to control the dynamics (6). The boundary ports are then defined by

$$\begin{aligned} u_s &:= \gamma_0(T), \quad y_s := -\gamma_\perp(J_s), \\ \widetilde{u}_r &:= \Gamma - \gamma_\perp(\rho v), \quad \widetilde{y}_r := \gamma_0(u). \end{aligned} \tag{16}$$

Hence (15) reads

$$\mathrm{d}_t\mathcal{H}_u(\overline{s},\rho) = \langle \widetilde{u}_r, \widetilde{y}_r\rangle_{\mathbb{R}^2} + \langle u_s, y_s\rangle_{\mathbb{R}^2} + \langle \partial_x v, \sigma\rangle_{L^2}. \tag{17}$$

3.3 An underlying Dirac structure

One can define the form of a state solution $(\overrightarrow{\boldsymbol{f}}_u, \overrightarrow{\boldsymbol{e}}_u)$ as follows

$$\begin{aligned}\overrightarrow{\boldsymbol{f}}_u &:= (\partial_t\overline{s},\ \partial_t\rho,\ -\partial_x T,\ -\partial_x T,\ -T,\ -\partial_x u,\ u_s,\ -\widetilde{y}_r)^\top, \\ \overrightarrow{\boldsymbol{e}}_u &:= (T,\ u,\ \overline{s}v,\ J_s,\ \Sigma,\ \rho v,\ -y_s,\ \widetilde{u}_r)^\top,\end{aligned}$$

and the *formal* associated structure operator

$$\mathcal{J}_u := \begin{pmatrix} 0 & 0 & -\partial_x & -\partial_x & 1 & 0 & \gamma_0^* & 0 \\ 0 & 0 & 0 & 0 & 0 & -\partial_x & 0 & \gamma_0^* \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\partial_x & 0 & 0 & 0 & 0 & 0 & 0 \\ -\gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

Theorem 3. With the domain $\mathcal{E}_1^u := \left(H^1(a,b)\right)^6 \times \left(\mathbb{R}^2\right)^2$, the effort space $\mathcal{E}^u := \left(L^2(a,b)\right)^6 \times \left(\mathbb{R}^2\right)^2$, the flow space $\mathcal{F}^u := (\mathcal{E}^u)' = \left(L^2(a,b)\right)^6 \times \left(\mathbb{R}^2\right)^2$, and the Bond space $\mathcal{B}^u := \mathcal{F}^u \times \mathcal{E}^u$ endowed with the symmetrized product (1), $\mathcal{D}^u := \mathrm{Graph}\,(\mathcal{J}_u)$ is an underlying Dirac structure for (14).

The proof is similar to that of Theorem 2.

4. THE PISTON PROBLEM

Now we interconnect the six sub-systems (two energies per physical domain) to get the full model of the piston.

4.1 Interconnection of kinetic and internal energies

In this section, we are interested in one physical domain only, but we want to model the variation of the total energy $\mathcal{H}(p,\overline{s},\rho) := \mathcal{H}_k(p,\rho) + \mathcal{H}_u(\overline{s},\rho)$. Obviously, from (10)–(15)

$$\begin{aligned}\mathrm{d}_t\mathcal{H}(p,\overline{s},\rho) &= \mathrm{d}_t\mathcal{H}_k(p,\rho) + \mathrm{d}_t\mathcal{H}_u(\overline{s},\rho) \\ &= \rho(t,b(t))\,(v(t,b(t)) - \mathrm{d}_t b(t))\left(-\frac{v^2(t,b(t))}{2} - u(t,b(t))\right) \\ &\quad -\rho(t,a(t))\,(v(t,a(t)) - \mathrm{d}_t a(t))\left(-\frac{v^2(t,a(t))}{2} - u(t,a(t))\right) \\ &\quad + v(t,b(t))\sigma(t,b(t)) - v(t,a(t))\sigma(t,a(t)) \\ &\quad - J_q(t,b(t)) + J_q(t,a(t)).\end{aligned}$$

or with the definitions of the boundary ports (11)–(16), noting that $u_r = -\widetilde{u}_r$ in (12)–(17)

$$\mathrm{d}_t\mathcal{H}(p,\overline{s},\rho) = \langle u_\sigma, y_\sigma\rangle_{\mathbb{R}^2} + \langle u_r, y_r - \widetilde{y}_r\rangle_{\mathbb{R}^2} + \langle u_s, y_s\rangle_{\mathbb{R}^2}.$$

Remark that the total energy is preserved for the closed system, as expected. Also note that $\widetilde{y}_r - y_r = \gamma_0(u) + \gamma_0\left(\frac{v^2}{2}\right)$ is nothing but the boundary trace of the total energy density.

The composition of the Dirac structures $\mathcal{D}^k$ and $\mathcal{D}^u$ is performed by taking into account the equalities of the common flows and efforts in the sub-systems modelling the kinetic and the internal energies. It is then required to set $u_r + \widetilde{u}_r = 0$, $\partial_t\rho_k = \partial_t\rho_u$ and $(\rho v)_k = (\rho v)_u$, where the subscript k (resp. u) means that the flow/effort belongs to the sub-system for $\mathcal{H}_k$ (resp. $\mathcal{H}_u$).

Let $\mathcal{C} := \left(L^2(a,b)\right)^2 \times \mathbb{R}^2$ and define the interconnection operators

$$C_k := \begin{pmatrix} 0, & I_{L^2}, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & I_{L^2}, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & I_2 \end{pmatrix}^\top \in \mathcal{L}\left(\mathcal{C}, \mathcal{F}^k\right),$$

$$C_u := \begin{pmatrix} 0, & -I_{L^2}, & 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0, & -I_{L^2}, & 0 \\ 0, & 0, & 0, & 0, & 0, & 0, & I_2 \end{pmatrix}^\top \in \mathcal{L}\left(\mathcal{C}, \mathcal{F}^u\right),$$

where I_{L^2} is the identity operator in $L^2(a,b)$ and I_2 the identity matrix in $\mathbb{R}^2$.
Hence, defining the Bond space $\mathcal{B} := \mathcal{E} \times \mathcal{F}$, where

$$\mathcal{E} := \mathcal{E}^k \times \mathcal{E}^u \times \mathcal{C}, \qquad \mathcal{F} := \mathcal{F}^k \times \mathcal{F}^u \times \mathcal{C},$$

and the (formal) structure operator $\mathcal{J} := \begin{pmatrix} \mathcal{J}_k & 0 & -C_k \\ 0 & \mathcal{J}_u & -C_u \\ C_k^* & C_u^* & 0 \end{pmatrix}$, the graph of $\mathcal{J}$, which proves to be in $\mathcal{L}\left(\mathcal{E}_1^k \times \mathcal{E}_1^u, \mathcal{F}\right)$ and such that $\left\langle \mathcal{J}\overrightarrow{\boldsymbol{e}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}} = -\left\langle \mathcal{J}\widetilde{\overrightarrow{\boldsymbol{e}}}, \overrightarrow{\boldsymbol{e}} \right\rangle_{\mathcal{F},\mathcal{E}}$ for all $\overrightarrow{\boldsymbol{e}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \in \mathcal{E}_1^k \times \mathcal{E}_1^u$, is an underlying Dirac structure for $\mathcal{H}_{\text{Tot}}$. The state solutions of interest are then of the form

$$\overrightarrow{\boldsymbol{f}}_s := \left(\overrightarrow{\boldsymbol{f}}_k,\ \overrightarrow{\boldsymbol{f}}_u,\ 0,\ 0,\ 0\right)^\top,\ \overrightarrow{\boldsymbol{e}}_s := \left(\overrightarrow{\boldsymbol{e}}_k,\ \overrightarrow{\boldsymbol{e}}_u,\ \frac{v^2}{2} + u,\ 0,\ 0\right)^\top.$$

Remark 5. Note that a Lagrange multiplier is required to be able to construct the Dirac structure $\mathcal{D}_{\text{Tot}}$ (this is a *transformer interconnection*). The value of this Lagrange multiplier is easily identified as the total energy density in the coupled system: $\frac{v^2}{2} + u$.

4.2 Interconnection between physical domains

Let S_1, S_2 and S_3 be the three physical domains $(a_0(t), a_1(t)) = (0, a_1(t))$, $(a_1(t), a_2(t))$ and $(a_2(t), a_3(t)) = (a_2(t), L)$, associated to the three total energies $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ respectively (and with 1, 2 or 3 superscript for the associated variables). Then the total energy in the piston is given by $\mathcal{H}_{\text{Tot}} := \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$. Assuming that the whole piston system is at thermal and mechanical equilibria and that no matter exchange between distinct sub-systems is to be found, we have at the internal interfaces

$$J_q^1(t, a_1(t)) = -J_q^2(t, a_1(t)), \quad J_q^2(t, a_2(t)) = -J_q^3(t, a_2(t)),$$
$$\sigma^1(t, a_1(t)) = -\sigma^2(t, a_1(t)), \quad \sigma^2(t, a_2(t)) = -\sigma^3(t, a_2(t)).$$

Furthermore, the internal moving boundaries are only driven by the velocity of the matter, hence one also has

$$v^1(t, a_1(t)) = v^2(t, a_1(t)) = \mathrm{d}_t a_1(t),$$
$$v^2(t, a_2(t)) = v^3(t, a_2(t)) = \mathrm{d}_t a_2(t).$$

Newton's second law is used for the boundary values of ρ^i. This leads to the following power balance for the total system

$$\mathrm{d}_t \mathcal{H}_{\text{Tot}}(t) = \rho^1(t,0) v^1(t,0) \left(\frac{(v^1)^2(t,0)}{2} + u(t,0) \right) - \sigma^1(t,0) v^1(t,0) - J_q^1(t,0) + J_q^3(t,L). \quad (19)$$

Following the same strategy as in the previous section, one can again define an underlying Dirac structure for $\mathcal{H}_{\text{Tot}}$ with these equilibrium hypotheses and adherence conditions, indeed leading to equalities of boundary ports.

4.3 Closure equations

As usual in PDEs, boundary conditions (boundary ports) have to be imposed. Some are imposed by physical considerations, others are left free, as inputs. In a_0 (supposed fixed in 0), the left end condition of pressure (σ^1), rate of flow ($\rho^1 v^1$) and heating (J_q^1) are used to control the piston. At the other end a_3 (supposed fixed in L), the velocity is supposed to be 0. The heat flux can be imposed *e.g.* as an exchange law with the environment (such as Newton's law of cooling). These controls dictate the available colocated observations, as expected for a pHs modelling.

As already mentionned in the introduction, if a solution exists, it is not necessarily unique (even with the above boundary conditions). For the time being, comportemental laws have not been taken into account, and each subsystem could be a gas, a liquid or a solid: σ^i have to be defined. The same remark holds for the heat fluxes J_q^i. These assumptions are called *constitutive relations* and link flows and efforts together.

Some of them have already been given as definitions. Such that $p := \rho v$, $\overline{u} := \rho u$, $\overline{s} := \rho s$, Σ defined by (7), the inputs and outputs, etc. Nervetheless, the system is not closed yet. Gibbs' formula (5) links u and s but introduces a new variable: the pressure π. We can close the system to model a simple 1D piston by adding the following laws: fluids are Newtonians, *i.e.* $\sigma^i := \nu^i \partial_x v^i - \pi^i$ for $i = 1, 3$, where ν^i is the kinematic viscosity. The fluid temperatures follow an ideal gas law: $\pi^i(a_i - a_{i-1}) = n^i R T^i$ where n^i is the amount of substance in fluid i for $i = 1, 2$ and R is the universal gas constant. The solid is rigid, *i.e.* $\sigma^2 \equiv 0$. And finally, the heat fluxes are driven by Fourier's law $J_q^i := -\lambda^i \partial_x T^i$, for $i = 1, 2, 3$, where λ^i are the heat conductivity. Finally, Dulong-Petit's model can be used to get that $\underline{u}^i = C_v^i \rho^i T^i$ where C_v^i is the isochoric heat capacity.

Altogether, we have enough equations at our disposal. The piston system then reads

$$\begin{cases} \partial_t \rho^1 = -\partial_x(p^1), \\ \partial_t p^1 = -\partial_x(p^1 v^1) + \partial_x\left(\nu^1 \partial_x(v^1) - \dfrac{n^1 R T^1}{a_1}\right), \\ C_v^1 \partial_t(\rho^1 T^1) = -\partial_x(C_v^1 \rho^1 T^1) \\ \qquad + \left(\nu^1 \partial_x(v^1) - \dfrac{n^1 R T^1}{a_1}\right) \partial_x(v^1) + \partial_x(\lambda^1 \partial_x(T^1)), \\ \rho^2 \partial_t v^2 = -(v^2)^2 \partial_x(\rho^2), \\ C_v^2 \rho^2 \partial_t(T^2) = -\partial_x(C_v^2 \rho^2 T^2) + \partial_x(\lambda^2 \partial_x(T^2)), \\ \partial_t \rho^3 = -\partial_x(p^3) \\ \partial_t p^3 = -\partial_x(p^3 v^3) + \partial_x\left(\nu^3 \partial_x(v^3) - \dfrac{n^3 R T^3}{L - a_2}\right), \\ C_v^3 \partial_t(\rho^3 T^3) = -\partial_x(C_v^3 \rho^3 T^3) \\ \qquad + \left(\nu^3 \partial_x(v^3) - \dfrac{n^3 R T^3}{L - a_2}\right) \partial_x\left(v^3\right) + \partial_x(\lambda^3 \partial_x(T^3)), \end{cases}$$

together with the collected boundary conditions. This model can be compared to those often used, as in the references cited in Section 1.

The advantage here is that the modelling is more versatile, since comportemental laws have been postponed as far as possible. In particular, an underlying Dirac structure $\mathcal{D}_{\text{Tot}}$ *encoding* the power balance (19) has been constructed independently of these comportemental laws.

5. CONCLUSION AND PERSPECTIVES

In this work, a model of the boundary-controlled 1D piston problem has been proposed, as the interconnection of six port-Hamiltonian systems with thermodynamics, but no chemical reaction. A major drawback, common for infinite dimensional port-Hamiltonian systems theory, is that no existence and uniqueness theorem is to be found in full generality. The strenghts of this strategy are the physically meaningful controls and the postponement of most of the physical hypotheses. The proposed model also applies for *e.g.* : (non-)linear elastic body (changing the constitutive relation for the stress tensor σ^2), non-Newtonian fluids (changing σ^i for $i = 1, 3$), Cattaneo's law (changing J_q^i for $i = 1, 2, 3$), van der Waals' law (changing π^i for $i = 1, 3$), etc. State equations are taken into account "outside" the Dirac structure $\mathcal{D}_{\text{Tot}}$, as constraints.

Theoretically, two perspectives will be investigated: model the piston in 2D, and add chemical reactions (following *e.g.* Altmann and Schulze (2017)).

Another interesting further work would be the development of a structure-preserving numerical scheme. A recent method, known as PFEM for Partitioned Finite Element Method, suits well for static geometrical domain (see for instance Cardoso-Ribeiro et al. (2019); Serhani et al. (2019b)), and seems to be an accurate starting point. The difficult part is to properly (*i.e.* by preserving the structure) take into account the moving boundaries.

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Appendix A. PROOF OF THEOREM 1

Proof. Let $\left(\overrightarrow{\boldsymbol{f}},\ \overrightarrow{\boldsymbol{e}}\right) \in \mathcal{D} := \text{Graph}\left(\mathcal{J}\right)$. Then for all $\left(\widetilde{\overrightarrow{\boldsymbol{f}}},\ \widetilde{\overrightarrow{\boldsymbol{e}}}\right) \in \mathcal{D}$, one has

$$\begin{aligned}
&\left[\left(\overrightarrow{\boldsymbol{f}},\ \overrightarrow{\boldsymbol{e}}\right), \left(\widetilde{\overrightarrow{\boldsymbol{f}}},\ \widetilde{\overrightarrow{\boldsymbol{e}}}\right)\right]_{\mathcal{B}} \\
&= \left\langle \overrightarrow{\boldsymbol{f}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \widetilde{\overrightarrow{\boldsymbol{f}}}, \overrightarrow{\boldsymbol{e}} \right\rangle_{\mathcal{F},\mathcal{E}}, \\
&= \left\langle \mathcal{J}\overrightarrow{\boldsymbol{e}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \mathcal{J}\widetilde{\overrightarrow{\boldsymbol{e}}}, \overrightarrow{\boldsymbol{e}} \right\rangle_{\mathcal{F},\mathcal{E}}, \\
&= \left\langle \mathcal{J}\left(\overrightarrow{\boldsymbol{e}} - \widetilde{\overrightarrow{\boldsymbol{e}}}\right), \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}} - \left\langle \mathcal{J}\left(\overrightarrow{\boldsymbol{e}} - \widetilde{\overrightarrow{\boldsymbol{e}}}\right), \overrightarrow{\boldsymbol{e}} \right\rangle_{\mathcal{F},\mathcal{E}}, \\
&= -\left\langle \mathcal{J}\left(\widetilde{\overrightarrow{\boldsymbol{e}}} - \overrightarrow{\boldsymbol{e}}\right), \widetilde{\overrightarrow{\boldsymbol{e}}} - \overrightarrow{\boldsymbol{e}} \right\rangle_{\mathcal{F},\mathcal{E}}, \\
&= 0.
\end{aligned}$$

This shows that $\mathcal{D} \subset \mathcal{D}^{[\perp]}$, *i.e.* $\mathcal{D}$ is a Tellegen structure.

Conversely, let $\left(\overrightarrow{\boldsymbol{f}},\ \overrightarrow{\boldsymbol{e}}\right) \in \mathcal{D}^{[\perp]}$. Then for all $\widetilde{\overrightarrow{\boldsymbol{e}}} \in \mathcal{E}_1$, *i.e.* for all $\left(\mathcal{J}\widetilde{\overrightarrow{\boldsymbol{e}}},\ \widetilde{\overrightarrow{\boldsymbol{e}}}\right) \in \mathcal{D}$, and all sequences $\overrightarrow{\boldsymbol{e}}_n \in \mathcal{E}_1 \to \overrightarrow{\boldsymbol{e}}$ in $\mathcal{E}$, one has

$$\begin{aligned}
0 &= \left[\left(\overrightarrow{\boldsymbol{f}},\ \overrightarrow{\boldsymbol{e}}\right), \left(\mathcal{J}\widetilde{\overrightarrow{\boldsymbol{e}}},\ \widetilde{\overrightarrow{\boldsymbol{e}}}\right)\right]_{\mathcal{B}} \\
&= \left\langle \overrightarrow{\boldsymbol{f}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}} + \left\langle \mathcal{J}\widetilde{\overrightarrow{\boldsymbol{e}}}, \lim_{n\to\infty} \overrightarrow{\boldsymbol{e}}_n \right\rangle_{\mathcal{F},\mathcal{E}}, \\
&= \left\langle \overrightarrow{\boldsymbol{f}}, \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}} - \lim_{n\to\infty} \left\langle \mathcal{J}\overrightarrow{\boldsymbol{e}}_n, \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}}, \\
&= \lim_{n\to\infty} \left\langle \overrightarrow{\boldsymbol{f}} - \mathcal{J}\overrightarrow{\boldsymbol{e}}_n, \widetilde{\overrightarrow{\boldsymbol{e}}} \right\rangle_{\mathcal{F},\mathcal{E}}.
\end{aligned}$$

This holds for all $\widetilde{\overrightarrow{\boldsymbol{e}}} \in \mathcal{E}_1$, which is dense in $\mathcal{X}$, hence the above limit is weakly-star convergent to 0. Since $\mathcal{J}$ is closed, it holds $\overrightarrow{\boldsymbol{f}} = \mathcal{J}\overrightarrow{\boldsymbol{e}}$, *i.e.* $\left(\overrightarrow{\boldsymbol{f}},\ \overrightarrow{\boldsymbol{e}}\right) \in \mathcal{D}$. ■