A Partitioned Finite Element Method for the Structure-Preserving Discretization of Damped Infinite-Dimensional Port-Hamiltonian Systems with Boundary Control

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Abstract. Many boundary controlled and observed Partial Differential Equations can be represented as port-Hamiltonian systems with dissipation, involving a Stokes-Dirac geometrical structure together with constitutive relations. The Partitioned Finite Element Method, introduced in Cardoso-Ribeiro et al. (2018), is a structure preserving numerical method which defines an underlying Dirac structure, and constitutive relations in weak form, leading to finite-dimensional port-Hamiltonian Differential Algebraic systems (pHDAE). Different types of dissipation are examined: internal damping, boundary damping and also diffusion models.

Keywords: Port-Hamiltonian systems · Dissipation · Structure preserving method · Partitioned Finite Element Method

1 Introduction

In this work, we are interested in infinite-dimensional dynamical systems representing *open* physical systems, *i.e.* with control v_∂ and observation y_∂ located at the boundary $\partial\Omega$ of the geometrical domain $\Omega \subset \mathbb{R}^d$. When the corresponding closed physical system proves conservative w.r.t. a given Hamiltonian functional H, the open system is said to be *lossless*. When it proves dissipative, the open system is said to be *lossy*. Here we use the port-Hamiltonian formalism, introduced a few decades ago, see e.g. [9,17,21,22]. Note that very different multi-physics applications can be described through it, e.g. plasmas in tokamaks [27], or fluid structure interaction [6]. The underlying geometry of the dynamical systems relies on a so-called Stokes-Dirac structure, see [8]; for the system to be

This work has been performed in the frame of the Collaborative Research DFG and ANR project INFIDHEM, entitled *Interconnected Infinite-Dimensional systems for Heterogeneous Media*, n° ANR-16-CE92-0028. Further information is available at https://websites.isae-supaero.fr/infidhem/the-project/.

F. Nielsen and F. Barbaresco (Eds.): GSI 2019, LNCS 11712, pp. 549–558, 2019.
https://doi.org/10.1007/978-3-030-26980-7_57

well-defined, some constitutive equations have to be added to the geometrical structure.

Our main concern is to provide a numerical method that preserves, at the discrete level, the geometrical structure of the original controlled PDE; for short, we look for a structure-preserving method which automatically transforms the Stokes-Dirac structure into a finite-dimensional Dirac structure: in the last decade, quite a number of ways have been explored, see e.g. [10,13,19,20,26]. Recently in [4], a method based on the weak formulation of the Partial Differential Equation and the use of the celebrated Finite Element Method has emerged. One of its many advantages is the preservation of the geometrical structure. It has successfully been applied to 1-D and also n-D systems, linear and nonlinear systems, with uniform or space-varying coefficients; it enables to deal with scalar-valued fields, vector-valued fields and also tensor-valued fields. Wave equations are tackled in [4,25], Mindlin's or Kirchhoff's plate equations are considered in [2,3], the treatment of the shallow water equations together with a general presentation of the Partitioned Finite Element Method (PFEM) is to be found in [5]. However, only lossless open systems have been addressed up to now: thus, the present paper intends to enlarge the scope of PFEM to lossy open systems, based on dissipative closed systems. These can be nicely accounted for in the port-Hamiltonian framework by introducing specific interaction ports: resistive ports.

The paper is organized as follows: in Sect. 2 the structure preserving discretization procedure is presented on a damped wave equation (with both internal and boundary damping) by introduction of resistive ports, in Sect. 3 the extension is proposed to a diffusion model as another class of dissipative PDE. Conclusions are drawn and a few perspectives are given in Sect. 4.

2 A General Result of Structure-Preserving Discretization for Damped pHs

To fix ideas and notations, a simple 1-D PDE model borrowed from [28] is first recalled: the lossy transmission line, on domain $\Omega = (0, \ell)$.

Example 1: The Lossless Transmission Line. Let us choose as energy variables or state variables $q(z,t)$ the linear charge density, and $\varphi(z,t)$ the magnetic flux density. With uniform or space-varying coefficients $C(z)$ the distributed capacitance, and $L(z)$ the distributed inductance, let us define the Hamiltonian density $\mathcal{H}(q,\varphi) := \frac{1}{2}(\frac{q^2}{C} + \frac{\varphi^2}{L})$, and the Hamiltonian $H(q,\varphi) := \int_0^\ell \mathcal{H}(q,\varphi)\ \mathrm{d}z$. With a slight abuse of notation, $H(q(t),\varphi(t))$ will be denoted $H(t)$ in the sequel. The co-energy variables are defined as the variational derivatives of the Hamiltonian w.r.t. the energy variables: $u_C := \delta_q H = \frac{q}{C}$ is the voltage, and $i_L := \delta_\varphi H = \frac{\varphi}{L}$ is the current. The conservation laws for the lossless transmission line read $\partial_t q = -\partial_z i_L$ and $\partial_t \varphi = -\partial_z u_C$. This can be rewritten in vector

form $\partial_t \overrightarrow{\boldsymbol{X}} = \mathcal{J}\, \delta_{\overrightarrow{\boldsymbol{X}}} H$, or in a more compact and abstract form:

$$\overrightarrow{\boldsymbol{f}} = \mathcal{J}\, \overrightarrow{\boldsymbol{e}}, \quad \text{with } \mathcal{J} = \begin{pmatrix} 0 & -\partial_z \\ -\partial_z & 0 \end{pmatrix},$$

$$\text{and } \overrightarrow{\boldsymbol{X}} = \begin{pmatrix} q \\ \varphi \end{pmatrix}, \quad \overrightarrow{\boldsymbol{e}} := \delta_{\overrightarrow{\boldsymbol{X}}} H = \begin{pmatrix} e_q \\ e_\varphi \end{pmatrix} = \begin{pmatrix} u_C \\ i_L \end{pmatrix}, \quad \overrightarrow{\boldsymbol{f}} = \partial_t \overrightarrow{\boldsymbol{X}} = \begin{pmatrix} \partial_t q \\ \partial_t \varphi \end{pmatrix}.$$

$\overrightarrow{\boldsymbol{e}}$ are the effort, or co-energy variables, $\overrightarrow{\boldsymbol{f}}$ are the flows, and $\mathcal{J}$ the interconnection matrix. It is easy to prove that $\mathcal{J}$ is a formally skew-symmetric differential operator on $L^2(0,\ell;\mathbb{R}^2)$. This results in a conservative closed system: indeed $d_t H = \int_0^\ell \partial_t \overrightarrow{\boldsymbol{X}} \cdot \delta_{\overrightarrow{\boldsymbol{X}}} H = \int_0^\ell \overrightarrow{\boldsymbol{X}} \cdot \mathcal{J} \overrightarrow{\boldsymbol{X}} = (\overrightarrow{\boldsymbol{f}}, \overrightarrow{\boldsymbol{e}}) = 0$, with the usual scalar product in $L^2(0,\ell;\mathbb{R}^2)$.

For the study of the open system, we need to introduce boundary ports at the boundary $\partial\Omega = \{0\}\times\{\ell\}$, such as $e_\partial := \big(e_q(0),\, e_q(\ell)\big)^\top$, $f_\partial := \big(e_\varphi(0),\, -e_\varphi(\ell)\big)^\top$. With $(\overrightarrow{\boldsymbol{e}}, e_\partial) \in \mathcal{E}$ the effort space and $(\overrightarrow{\boldsymbol{f}}, f_\partial) \in \mathcal{F}$ the flow space, we define the bond space $\mathcal{B} := \mathcal{E} \times \mathcal{F}$, and introduce a bilinear product on $\mathcal{B}$, namely:

$$<(\overrightarrow{\boldsymbol{e}}, e_\partial), (\overrightarrow{\boldsymbol{f}}, f_\partial)> := \int_0^\ell \overrightarrow{\boldsymbol{e}} \cdot \overrightarrow{\boldsymbol{f}}\ \mathrm{d}z + (e_\partial, f_\partial)_{\mathbb{R}^2}.$$

A Dirac structure is a subset $\mathcal{D} \subset \mathcal{B}$ which is maximally isotropic w.r.t. the symmetrized product on $\mathcal{B}\times\mathcal{B}$, $<<(e_1,f_1),(e_2,f_2)>> := <e_1,f_2> + <e_2,f_1>$.

Proposition 1. *The subspace:*

$$\mathcal{D} := \{(e,f) \in \mathcal{B} \quad | \ \overrightarrow{\boldsymbol{f}} = \mathcal{J}\, \overrightarrow{\boldsymbol{e}},\ e_\partial := \big(e_q(0),\, e_q(\ell)\big)^\top,\ f_\partial := \big(e_\varphi(0),\, -e_\varphi(\ell)\big)^\top\},$$

is indeed a Stokes-Dirac structure.

As a consequence, the former conservative property of the closed system now generalizes into the following losslessness property for the open system:

$$d_t H(t) = -(e_\partial(t), f_\partial(t))_{\mathbb{R}^2}.$$

Example 2: The Lossy Transmission Line. Taking into account some losses with $R(z)$ the distributed resistance coefficient, leads to a new balance equation: $\partial_t q = -\partial_z i_L$ and $\partial_t \varphi = -\partial_z u_C - R i_L$. This can be first seen as a pHs with dissipation: $\partial_t \overrightarrow{\boldsymbol{X}} = (\mathcal{J} - \mathcal{R})\, \delta_{\overrightarrow{\boldsymbol{X}}} H$ with some positive symmetric bounded operator $\mathcal{R}$, implying the dissipativity of the closed system: indeed, $d_t H = \int_0^\ell \partial_t \overrightarrow{\boldsymbol{X}} \cdot \delta_{\overrightarrow{\boldsymbol{X}}} H = \int_0^\ell \delta_{\overrightarrow{\boldsymbol{X}}} H \cdot (\mathcal{J} - \mathcal{R})\, \delta_{\overrightarrow{\boldsymbol{X}}} H = -(\overrightarrow{\boldsymbol{e}}, \mathcal{R}\, \overrightarrow{\boldsymbol{e}}) \le 0$.

But the construction of a Stokes-Dirac structure associated to it requires the definition of extra resistive ports (e_R, f_R) which will now be related by an extra *constitutive relation* $e_R = R\, f_R$. Let us consider the natural extension $\overrightarrow{\boldsymbol{f}}_e = \mathcal{J}_e\, \overrightarrow{\boldsymbol{e}}_e$, where $\overrightarrow{\boldsymbol{e}}_e = (\overrightarrow{\boldsymbol{e}}, e_R)$, $\overrightarrow{\boldsymbol{f}}_e = (\overrightarrow{\boldsymbol{f}}, f_R)$ and the extended interconnection operator:

$$\mathcal{J}_e = \begin{pmatrix} 0 & -\partial_z & 0 \\ -\partial_z & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

With the extended bilinear product:

$$<(\overrightarrow{e}, e_\partial, e_R), (\overrightarrow{f}, f_\partial, f_R)> := \int_0^\ell (\overrightarrow{e} \cdot \overrightarrow{f} + e_R\, f_R)\ \mathrm{d}z + (e_\partial, f_\partial)_{\mathbb{R}^2},$$

a new Stokes-Dirac structure can be defined. As a consequence, thanks to $e_R = R\, f_R$, the former dissipative property of the closed system now generalizes into the following lossy property for the open system:

$$d_t H(t) = -\int_0^\ell R\, f_R^2(t) - (e_\partial(t), f_\partial(t))_{\mathbb{R}^2} \leq -(e_\partial(t), f_\partial(t))_{\mathbb{R}^2}.$$

Note that for the dissipative system to be correctly defined, one actually needs an extra constitutive relation to close the system. In fact, we have:

$$\begin{pmatrix} \partial_t q \\ \partial_t \varphi \\ f_R \end{pmatrix} = \begin{pmatrix} 0 & -\partial_z & 0 \\ -\partial_z & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_q \\ e_\varphi \\ e_R \end{pmatrix}.$$

The first two lines are dynamical equations (once the link between the efforts $\overrightarrow{e}$ and the state variables $\overrightarrow{X}$, called a *constitutive relation*, has been made explicit: in the present case it is a diagonal linear transform) which must be complemented by initial data, while the third line is an algebraic equation, to which a *closure equation* must be added, namely $e_R = R\, f_R$.

PFEM consists of two steps: the definition of the Dirac structure from the original Stokes-Dirac structure in Sect. 2.1, and the definition of the constitutive relations at the discrete level in Sect. 2.2. Both steps are now detailed on the n-D case of a wave equation with internal damping, see e.g. [18].

Let us consider the damped wave equation of the form:

$$\rho(\mathbf{x})\, \partial_{tt}^2 w(t,\mathbf{x}) + \epsilon(\mathbf{x})\, \partial_t w(t,\mathbf{x}) = \mathrm{div}\left(\overline{\overline{T}}(\mathbf{x}) \cdot \overrightarrow{\mathbf{grad}}\, w(t,\mathbf{x})\right), \quad \mathbf{x} \in \Omega,$$

with $\epsilon \geq 0$. Define as energy variables the strain $\boldsymbol{\alpha}_q(t,\mathbf{x}) := \overrightarrow{\mathbf{grad}}\, w(t,\mathbf{x})$, and the linear momentum $\alpha_p(t,\mathbf{x}) := \rho(\mathbf{x}) \partial_t w(t,\mathbf{x})$. Taking the mechanical energy as Hamiltonian $H(t) := \frac{1}{2} \int_\Omega \boldsymbol{\alpha}_q(t,\mathbf{x})^\top \cdot \overline{\overline{T}}(\mathbf{x}) \cdot \boldsymbol{\alpha}_q(t,\mathbf{x}) + \frac{1}{\rho(\mathbf{x})} \alpha_p(t,\mathbf{x})^2\ \mathrm{d}\mathbf{x}$, the corresponding co-energy variables are the stress $\boldsymbol{e}_q := \delta_{\boldsymbol{\alpha}_q} \mathcal{H} = \overline{\overline{T}} \cdot \boldsymbol{\alpha}_q$, and the velocity $e_p := \delta_{\alpha_p} \mathcal{H} = \frac{1}{\rho} \alpha_p$. Introducing damping ports, the PDE can be written:

$$\begin{bmatrix} \partial_t \boldsymbol{\alpha}_q \\ \partial_t \alpha_p \\ f_r \end{bmatrix} = \begin{bmatrix} 0 & \overrightarrow{\mathbf{grad}} & 0 \\ \mathrm{div} & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_q \\ e_p \\ e_r \end{bmatrix}, \tag{1}$$

together with the closure relation $e_r = \epsilon\, f_r$.

As seen in Example 1, boundary ports can be taken as traces of the efforts. Let us then denote $\overrightarrow{\mathbf{n}}$ the outward normal to Ω, and define the boundary ports:

$$u_\partial := e_p\big|_{\partial\Omega}, \qquad y_\partial := \overrightarrow{e}_q \cdot \overrightarrow{\mathbf{n}}.$$

This also gives rise to a Stokes-Dirac structure (thanks to Green's formula), and taking $e_\partial := u_\partial$ and $f_\partial := -y_\partial$, one immediately has the following lossy property:

$$d_t H(t) = -\int_0^\ell \epsilon\, f_r^2(t) - (e_\partial(t), f_\partial(t))_{\partial\Omega} \leq (u_\partial(t), y_\partial(t))_{\partial\Omega}. \tag{2}$$

Impedance Boundary Conditions (IBC) can easily be taken into account within this formalism: for $\mathbf{x} \in \partial\Omega$, let $Z(\mathbf{x}) \geq 0$ be the impedance, and take $u_\partial = -Zy_\partial + v_\partial$ as control, where v_∂ is an extra boundary control. Indeed, it means that the IBC $\partial_t w + Z\left(\overline{\overline{T}} \cdot \overrightarrow{\mathbf{grad}}(w)\right) \cdot \overrightarrow{\mathbf{n}} = v_\partial$ is imposed to the original system. The previous power balance then reads:

$$d_t H(t) = -\int_0^\ell \epsilon\, f_r^2(t) - (Zy_\partial(t), y_\partial(t))_{\partial\Omega} + (v_\partial(t), y_\partial(t))_{\partial\Omega} \leq (v_\partial(t), y_\partial(t))_{\partial\Omega}.$$

Note that, as it has been said in Example 2, and as it has been done above with the introduction of the resistive ports f_r and e_r, the construction of a Stokes-Dirac structure for the wave equation with IBC requires another extension of the interconnection operator, *i.e.* boundary resistive ports have to be added. However, this latter task is not that straightforward, since it involves unbounded trace operators.

2.1 Stokes-Dirac Structure Translates into a Dirac Structure

Let us write a weak form of (1) with $\boldsymbol{v}_q$ and v_p as test functions, and apply Green's formula to the first line only, to make the boundary control term appear, $u_\partial = e_p\big|_{\partial\Omega}$. Thus, we get $(\partial_t \boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega = -(e_p, \operatorname{div} \boldsymbol{v}_q)_\Omega + (u_\partial, \boldsymbol{v}_q \cdot \overrightarrow{\mathbf{n}})_{\partial\Omega}$ and $(\partial_t \alpha_p, v_p)_\Omega = (\operatorname{div} \boldsymbol{e}_q, v_p)_\Omega - (e_r, v_p)_\Omega$. Let us choose finite-element bases: $\boldsymbol{\alpha}_q^d(t, \mathbf{x}) := \Sigma_{i=1}^{N_q} \alpha_q^i(t) \boldsymbol{\varphi}_q^i(\mathbf{x}) = \boldsymbol{\Phi}_q^\top \cdot \underline{\alpha}_q(t)$ and similarly for $\boldsymbol{e}_q^d$ for the q vector-valued variables in the basis $\boldsymbol{\varphi}_q$; $\alpha_p^d(t, \mathbf{x}) := \Sigma_{k=1}^{N_p} \alpha_p^k(t) \varphi_p^k(\mathbf{x}) = \boldsymbol{\Phi}_p^\top \cdot \underline{\alpha}_p(t)$ and similarly for e_p^d for the p scalar-valued variables in the basis φ_p; $f_r^d(t, \mathbf{x}) := \Sigma_{k=1}^{N_p} f_r^k(t) \varphi_p^k(\mathbf{x}) = \boldsymbol{\Phi}_r^\top \cdot \underline{f_r}(t)$ and similarly for e_r^d for the r scalar-valued variables in the basis φ_r; and $u_\partial^d(t, \mathbf{x}) := \Sigma_{m=1}^{N_\partial} u_\partial^k(t) \psi_\partial^m(\mathbf{x}) = \boldsymbol{\Psi}_\partial^\top \cdot \underline{u}_\partial(t)$ and similarly for y_∂^d for the boundary variables in the basis ψ_∂. Plugging the finite-dimensional approximations into the above weak form leads to the following pHs:

$$\begin{cases} M_q\ d_t \underline{\alpha}_q = D\, \underline{e}_p + B \underline{u}_\partial, \\ M_p\ d_t \underline{\alpha}_p = -D^\top\, \underline{e}_q + G\, \underline{e}_r, \\ M_r\ \underline{f}_r = -G^\top\, \underline{e}_p, \\ M_\partial\, \underline{y}_\partial = B^\top \underline{e}_q, \end{cases} \tag{3}$$

with mass matrices $M_q = \int_\Omega \boldsymbol{\Phi}_q \cdot \boldsymbol{\Phi}_q^\top \in \mathbb{R}^{N_q \times N_q}$, $M_p = \int_\Omega \boldsymbol{\Phi}_p \cdot \boldsymbol{\Phi}_q^\top \in \mathbb{R}^{N_p \times N_p}$ and $M_\partial = \int_{\partial\Omega} \boldsymbol{\Psi}_\partial \cdot \boldsymbol{\Psi}_\partial^\top \in \mathbb{R}^{N_\partial \times N_\partial}$, a control matrix $B := \int_{\partial\Omega} \boldsymbol{\Phi}_q \cdot \overrightarrow{\mathbf{n}} \cdot \boldsymbol{\Psi}_\partial^\top \in \mathbb{R}^{N_q \times N_\partial}$,

and a structure matrix J composed of $D := -\int_\Omega \operatorname{div} \boldsymbol{\Phi}_q \cdot \boldsymbol{\Phi}_p^\top \in \mathbb{R}^{N_q \times N_p}$ and $G := \int_\Omega \boldsymbol{\Phi}_p \cdot \boldsymbol{\Phi}_r^\top \in \mathbb{R}^{N_p \times N_r}$. It is then straightforward to define a bilinear product on $\mathcal{B}^d := \mathbb{R}^{N_q+N_p+N_r+N_\partial} \times \mathbb{R}^{N_q+N_p+N_r+N_\partial}$ as $<(\underline{e}_q, \underline{e}_p, \underline{e}_r, \underline{e}_\partial), (\underline{f}_q, \underline{f}_p, \underline{f}_r, \underline{f}_\partial)> := \underline{e}_q^\top M_q \underline{f}_q + \underline{e}_p^\top M_p \underline{f}_p + \underline{e}_r^\top M_r \underline{f}_r + \underline{e}_\partial^\top M_\partial \underline{f}_\partial$.

Proposition 2. *The subspace:*

$$\mathcal{D}^d := \{(e, f) \in \mathcal{B}^d \,|\, (\underline{f}_q, \underline{f}_p, \underline{f}_r)^\top = J(\underline{e}_q, \underline{e}_p, \underline{e}_r)^\top,\ e_\partial := \underline{u}_\partial,\ f_\partial := -\underline{y}_\partial\},$$

is a Dirac structure.

Remark 1. Moreover, contrarily to other structure-preserving methods relying on Stokes-Dirac structure, like [13,20], there is no need here to project, reduce, some non square matrices in order to recover a full rank system at the discrete level, which is, at least from the numerical point of view, a severe limitation indeed.

2.2 Constitutive Relation Are Approximated in Weak Form

The idea is fairly simple: the constitutive equation of the resistive port $e_r = \epsilon f_r$ is written in weak form, and using the previously defined approximation f_r^d and e_r^d, one gets:

$$M_r\, \underline{e}_r(t) = <R>\, \underline{f}_r(t),$$

involving two symmetric $N_r \times N_r$ matrices, the mass matrix $M_r := \int_\Omega \boldsymbol{\Phi}_r^\top \boldsymbol{\Phi}_r \ \mathrm{d}\mathbf{x}$ which is positive definite, and $<R> := \int_\Omega \boldsymbol{\Phi}_r^\top \epsilon(\mathbf{x})\, \boldsymbol{\Phi}_r \ \mathrm{d}\mathbf{x}$, the averaged *resistive* matrix which is positive.

Finally, once these two steps have been carried out, we can prove the following

Proposition 3. *Defining the discrete Hamiltonian as:*

$$H_d(t) = H_d(\underline{\alpha}_q(t), \underline{\alpha}_p(t)) := H(\boldsymbol{\alpha}_q^d(t, \mathbf{x}), \alpha_p^d(t, \mathbf{x})),$$

the discrete counterpart of the continuous lossy property (2) holds for the finite-dimensional system (3) obtained with PFEM: $d_t H_d \leq \langle \underline{u}_\partial, \underline{y}_\partial \rangle_\partial := \underline{y}_\partial^\top M_\partial \underline{u}_\partial$.

Indeed, thanks to the Dirac structure and the constitutive relations, we have: $d_t H_d = \underline{e}_q^\top\ M_q\ d_t\underline{\alpha}_q + \underline{e}_p^\top\ M_p\ d_t\underline{\alpha}_p = -\underline{f}_r^\top <R> \underline{f}_r + \underline{y}_\partial^\top M_\partial \underline{u}_\partial \leq \langle \underline{u}_\partial, \underline{y}_\partial \rangle_\partial$.

Now, at the boundary, the IBC is discretized in the same manner above: let $<Z> := \int_{\partial\Omega} \boldsymbol{\Psi}_\partial^\top Z(\mathbf{x}) \boldsymbol{\Psi}_\partial \ \mathrm{d}\mathbf{x} \in \mathbb{R}^{N_\partial \times N_\partial}$ be the averaged resistive matrix taking Z into account on the boundary only. Then, define $v_\partial^d(t, \mathbf{x}) := \Sigma_{m=1}^{N_\partial} v_\partial^k(t) \psi_\partial^m(\mathbf{x}) = \boldsymbol{\Psi}_\partial^\top \cdot \underline{v_\partial}(t)$ the approximation of the extra control v_∂, and add to system (3) the following algebraic equation: $M_\partial \underline{u}_\partial(t) = -<Z> \underline{y}_\partial(t) + M_\partial \underline{v_\partial}(t)$, which mimicks $u_\partial = -Z y_\partial + v_\partial$ by finite element discretization on the boundary. We finally get:

$$d_t H_d = -\underline{f}_r^\top <R> \underline{f}_r - \underline{y}_\partial^\top <Z> \underline{y}_\partial + \underline{y}_\partial^\top M_\partial \underline{v_\partial} \leq \langle \underline{v_\partial}, \underline{y}_\partial \rangle_\partial.$$

Remark 2. At the continuous level, we have seen that the extension of the interconnection operator which gives rise to the Stokes-Dirac structure could be a difficult task, since it would involve unbounded operators (typically trace operators). However, once PFEM has been applied, it proves straightforward to define the resistive ports, both internal and at the boundary. Indeed we can write, with obvious notations:

$$\begin{pmatrix} M_q & 0 & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 & 0 \\ 0 & 0 & M_r & 0 & 0 \\ 0 & 0 & 0 & M_i & 0 \\ 0 & 0 & 0 & 0 & M_\partial \end{pmatrix} \begin{pmatrix} d_t\underline{\alpha}_q \\ d_t\underline{\alpha}_p \\ \underline{f_r} \\ \underline{f_i} \\ \underline{f_\partial} \end{pmatrix} = \begin{pmatrix} 0 & D & 0 & -B & B \\ -D^\top & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ B^\top & 0 & 0 & 0 & 0 \\ -B^\top & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{e_q} \\ \underline{e_p} \\ \underline{e_r} \\ \underline{e_i} \\ \underline{e_\partial} \end{pmatrix},$$

together with the two constitutive relations:

$$M_r\underline{e_r} = <R>\underline{f_r}, \qquad M_i\underline{e_i} = <Z>\underline{f_i},$$

and the definitions $\underline{f_\partial} = -\underline{y_\partial}$ and $\underline{e_\partial} = \underline{v_\partial}$ that are now usual in our approach. All together, the desired lossy property of the system is ensured at the discrete level.

Remark 3. Let us point out that the mass matrices on the left-hand side are required in order to preserve the underlying geometry. To some extent, they do discretize the bilinear form used to define the Stokes-Dirac structure, w.r.t. the chosen finite element families, as seen before in Proposition 2.

3 Diffusion Model in Dissipative Formulation

The heat or diffusion PDE is most often considered as a dissipative infinite-dimension system from a mathematical point of view, and examplifies the category of parabolic PDEs: this approach is being recalled here with the choice of a quadratic potential as Hamiltonian function, though its thermodynamical meaning is far from clear, see [23,24] for details and the choice of either energy or entropy as thermodynamically meaningful Hamiltonian.

Port-Hamiltonian System Model. Let $H(t) := \frac{1}{2}\int_\Omega \rho(\mathbf{x})\frac{(u(t,\mathbf{x}))^2}{C_V(t,\mathbf{x})}\,\mathrm{d}\mathbf{x}$ be the Hamiltonian with u the energy variable: ρ is the mass density, C_V is the isochoric heat capacity and u the internal energy density. The co-energy variable is $\delta_u\mathcal{H} = \frac{u}{C_V} = T$ (the temperature), assuming $u(t,\mathbf{x}) = C_V(\mathbf{x})\,T(t,\mathbf{x})$. Let us define $f_u := \partial_t u$, $e_u := T$ and $\overrightarrow{e}_Q := \overrightarrow{\mathbf{J}}_Q$ (the heat flux). Then, with $\overrightarrow{f}_Q := -\overrightarrow{\mathbf{grad}}(T)$, we get:

$$\begin{pmatrix} \rho f_u \\ \overrightarrow{f}_Q \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div} \\ -\overrightarrow{\mathbf{grad}} & 0 \end{pmatrix} \begin{pmatrix} e_u \\ \overrightarrow{e}_Q \end{pmatrix}.$$

The system must be completed by e.g. Fourier's law as constitutive relation:

$$\overrightarrow{\mathbf{J}}_Q(t,\mathbf{x}) = -\overline{\overline{\lambda}}(\mathbf{x}) \cdot \overrightarrow{\mathbf{grad}}\,(T(t,\mathbf{x}))\,, \quad \forall t \geq 0, \mathbf{x} \in \Omega, \tag{4}$$

where $\overline{\overline{\lambda}}$ is a tensor representing the thermal conductivity; it is a positive symmetric tensor thanks to Onsager's reciprocal relations.

For the boundary ports, one possible choice is $\mathcal{B}\overrightarrow{e} := e_{u|_{\partial\Omega}}$ the boundary temperature, and $\mathcal{C}\overrightarrow{e} := -(\overrightarrow{e}_Q \cdot \overrightarrow{\mathbf{n}})_{|_{\partial\Omega}}$ the incoming boundary flux. Hence, the power balance for this lossy open system is:

$$d_t H(t) = -\int_\Omega \overrightarrow{f}_Q(t,\mathbf{x}) \cdot \overline{\overline{\lambda}}(\mathbf{x}) \cdot \overrightarrow{f}_Q(t,\mathbf{x})\ \mathrm{d}\mathbf{x} + \int_{\partial\Omega} v_\partial(t,\gamma) y_\partial(t,\gamma)\ \mathrm{d}\gamma. \tag{5}$$

Partitioned Finite Element Method. Following the procedure explained in Sect. 2 and with obvious notations, we get:

$$\begin{cases} M_\rho\, \underline{f_u}(t) = D\, \underline{e_Q}(t) + B\, \underline{v_\partial}(t), \\ \overrightarrow{\boldsymbol{M}}\, \underline{f_Q}(t) = -D^\top\, \underline{e_u}(t), \\ M_\partial\, \underline{y_\partial}(t) = C\, \underline{e_u}(t), \end{cases}$$

where for example $D := \int_\Omega \overrightarrow{\mathbf{grad}}\,(\Phi) \cdot \overrightarrow{\boldsymbol{\Phi}}^\top\ \mathrm{d}\mathbf{x} \in \mathbb{R}^{N \times \overrightarrow{N}}$. The weak version of the constitutive law (4) reads:

$$\overrightarrow{\boldsymbol{M}}\underline{e_Q}(t) = \overrightarrow{\boldsymbol{\Lambda}}\underline{f_Q}(t), \quad \text{where}\, \overrightarrow{\boldsymbol{\Lambda}} := \int_\Omega \overrightarrow{\boldsymbol{\Phi}} \cdot \overline{\overline{\lambda}} \cdot \overrightarrow{\boldsymbol{\Phi}}^\top\ \mathrm{d}\mathbf{x} \in \mathbb{R}^{\overrightarrow{N} \times \overrightarrow{N}},$$

and thus the coupled system is now a pHDAE, see e.g. [1]; the energy balance (5) becomes at the discrete level:

$$d_t \mathcal{H}_d(t) = -\underline{f_Q}^\top(t)\, \overrightarrow{\boldsymbol{\Lambda}}\, \underline{f_Q}(t) + \underline{v_\partial}^\top(t) M_\partial \underline{y_\partial}(t).$$

4 Conclusion and Perspectives

In this paper, a structure-preserving numerical method has been presented for lossy port-Hamiltonian systems: the so-called Partitioned Finite Element Method (PFEM). It is based on the weak formulation of PDE, the application of a Stokes formula (reduced to Green formula in our examples) to get the useful boundary control explicitly, and the application of the classical finite element method with the choice of conforming elements for the different ports. Boundary damping, such as impedance boundary condition, studied theoretically in [15] as a pHs, becomes particularly straightforward with PFEM, since it results in a sparse damping matrix R at the discrete level (see [25] for more details).
The following perspectives seem relevant and promising:

– the choice of the finite element family remains quite open so far, but indeed, from first numerical experiments, some optimal choices can be observed in practice: the careful numerical analysis must still be investigated,

– structure-preserving model reduction can be carried out using methods presented in [11],
– for the time-domain discretization as last step procedure for numerical simulation, specific approaches should be followed, see e.g. [7],
– following [28], the introduction of entropy ports enables to transform dissipative systems into conservative systems, taking into account some thermodynamical laws; see [23,24] for the application to the heat equation.
– an alternative computational solution consists in making use of the transformation of the lossy system into a lossless one, and only then apply classical symplectic numerical schemes, see e.g. [12,16]; the difficulty lies in the fact that the obtained finite-dimensional systems are necessarily differential algebraic equations that should be treated with some specific care, see e.g. [14].

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