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Partitioned Finite Element Method for port-Hamiltonian systems with Boundary Damping: Anisotropic Heterogeneous 2D wave equations ⋆

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Abstract A $2D$ wave equation with boundary damping of impedance type can be recast into an infinite-dimensional port-Hamiltonian system (pHs) with an appropriate feedback law, where the structure operator $\mathcal{J}$ is formally skew-symmetric. It is known that the underlying semigroup proves dissipative, even though no dissipation operator $\mathcal{R}$ is to be found in the pHs model.
The Partitioned Finite Element Method (PFEM) introduced in Cardoso-Ribeiro et al. (2018), is structure-preserving and provides a natural way to discretize such systems. It gives rise to a non null symmetric matrix R. Moreover, since this matrix accounts for boundary damping, its rank is very low: only the basis functions at the boundary have an influence. Lastly, this matrix can be factorized out when considering the boundary condition as a feedback law for the pHs, involving the impedance parameter. Note that pHs – as open system – is used here as a tool to accurately discretize the wave equation with boundary damping as a *closed* system.
In the worked-out numerical examples in $2D$, the isotropic and homogeneous case is presented and the influence of the impedance is assessed; then, an anisotropic and heterogeneous wave equation with space-varying impedance at the boundary is investigated.

Keywords: Port-Hamiltonian systems (pHs), distributed-parameter system (DPS), structure preserving discretization, partitioned finite element method (PFEM), boundary damping.

1. INTRODUCTION

The analysis and modeling of multi-physical system as port-Hamiltonian systems (pHs) originates from Maschke and van der Schaft (1992), where the authors proposed a Hamiltonian formulation of lumped-parameter systems by introducing ports which account for energy exchange with the external environment. In van der Schaft and Maschke (2002), the authors provided a generalization of pHs formulation to distributed-parameter systems in a differential-geometrical setting. For an introductive overview, we refer the reader to e.g. van der Schaft and Jeltsema (2014); Duindam et al. (2009). Subsequently, several works have been carried out on the construction of a mathematical framework for linear infinite-dimensional pHs with the purpose of proving existence, uniqueness, and stability of the solutions, see e.g. Jacob and Zwart (2012).

In an attempt to perform numerical simulation or to apply some dedicated control laws, a growing body of structuring-preserving (SP) methods has been investigated in the last decade. By SP discretization, we mean a mimetic method that preserves the structure and all the properties of the infinite-dimensional pHs, when passing to the finite-dimensional system. In this direction, the authors in Golo et al. (2004) proposed a "mixed finite element method", also a pseudo-spectral method is provided for 1D pHs in in Moulla et al. (2012); a generalization of this method to arbitrary spatial dimension using mixed Galerkin discretization was recently proposed in Kotyczka et al. (2018).

A recent SP method, first introduced in Cardoso-Ribeiro et al. (2018), named Partitioned Finite Element Method (PFEM), is a kind of mixed finite element methods, based on the weak formulation of the port-Hamiltonian system, where an integration by parts is performed on a subset of the variables only. In this method, boundary control and observation appear naturally, which allows simulation and control of infinite-dimensional pHs. A major advantage of PFEM is that it can be easily implemented in a *Finite Element numerical software*, and does not require further specific treatment.

In this paper, we study the effect of boundary damping on a wave equation in 2D: the main parameter of interest is the impedance Z. The first systematic study and simulation of such a wave equation with an absorbing boundary condition was conducted in Engquist and Majda (1977), while in Cowsar et al. (1996), a mixed finite element method with different time integration schemes was fully studied. Otherwise, this paper presents an analysis and discretization of the dissipative pHs of anisotropic hetero-

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geneous wave equation with boundary damping. A class of dissipative pHs with structural and fluid damping was studied in Matignon and Hélie (2013) and Le Gorrec and Matignon (2013), while in Villegas et al. (2009) a boundary damping was used to get the exponential stability of a 1-D system. The theoretical analysis of our dissipative pHs heavily relies on the results of Kurula and Zwart (2015). Then, the discretization procedure is carried out in two steps: first, the pHs system with boundary control and observation is considered alone, and PFEM is applied to it, leading to collocated boundary control and observation of finite dimension; second, an output feedback control law is applied, at the discrete level, which mimicks the impedance boundary condition; moreover, in the last stage, a dissipation matrix R appears in a straighforward way. The motivation for such an approach relies on avoiding the explicit discretization of $D(\mathcal{J})$: indeed, since the boundary condition is natural (*i.e.* included in the domain of the unbounded operator), and not essential (*i.e.* included in the weak formulation of interest), one might think about adding some Lagrange multipliers (DAE) to overcome this difficulty, but this strategy proves irrelevant for such a problem. Finally, another advantage of this two-step approach is to provide a factorization of the dissipation matrix R giving meaningful information about its low rank, as shown in § 3.4.

The paper is organized as follows, *Section 2* reformulates the mathematical analysis of our wave equation based on results in Kurula and Zwart (2015). In *Section 3*, the discretization of the pHs formulation of the boundary controlled and observed wave equation is presented first, then the output feedback control law related to the impedance boundary condition is applied, the resulting matrices and their properties are given. *Section 4* is dedicated to the simulation results of two worked-out examples of the wave equation, as a closed system.

2. INFINITE-DIMENSIONAL PORT-HAMILTONIAN SYSTEM

2.1 Physical model at stake

We consider the 2-dimensional heterogeneous anisotropic wave equation defined for all $t \geq 0$ as

$$\begin{aligned}
&\rho(\mathbf{x})\,\partial^2_{tt} w(t,\mathbf{x}) = \operatorname{div}\left(\overline{\overline{T}}(\mathbf{x})\cdot\mathbf{grad}\, w(t,\mathbf{x})\right),\ \mathbf{x}\in\Omega,\\
&Z(\mathbf{x})\left(\overline{\overline{T}}(\mathbf{x})\cdot\mathbf{grad}\, w(t,\mathbf{x})\right)\cdot\mathbf{n} + \partial_t w(t,\mathbf{x}) = 0,\ \mathbf{x}\in\partial\Omega,\\
&w(0,\mathbf{x}) = w_0(\mathbf{x}),\ \mathbf{x}\in\Omega,\ t=0,\\
&\partial_t w(0,\mathbf{x}) = w_1(\mathbf{x}),\ \mathbf{x}\in\Omega,\ t=0,
\end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded spatial domain with Lipschitz-continuous boundary $\partial\Omega$, $w(t,\mathbf{x})$ is the deflection from the equilibrium position at point $\mathbf{x} \in \Omega$ and time $t \geq 0$, and vector $\mathbf{n}$ denotes the outward normal at the boundary. The physical parameters in the domain are $\rho \in L^\infty(\Omega)$ (positive and bounded from below) and $\overline{\overline{T}} \in L^\infty(\Omega)^{2\times 2}$ (symmetric and coercive), they denote the mass density and Young's elasticity modulus respectively. Finally, the physical parameter at the boundary, $Z(\mathbf{x}) \geq 0$, is the *impedance* function, the inverse of which is the *admittance*, accounting for boundary damping: the case where the admittance is null corresponds to a homogeneous non-standard Dirichlet boundary condition ($\partial_t w\big|_{\partial\Omega} = 0$, clamped), while when the impedance vanishes, we recover a homogeneous Neumann boundary condition for the same variable w (free).

2.2 Formulation as an infinite-dimensional Hamiltonian system

In order to formulate the wave equation with impedance boundary condition as a (closed) Hamiltonian system, we first introduce the energy variables $\boldsymbol{\alpha} := \begin{bmatrix}\boldsymbol{\alpha}_q^\top & \alpha_p\end{bmatrix}^\top$

$$\boldsymbol{\alpha}_q(t,\mathbf{x}) := \mathbf{grad}\, w(t,\mathbf{x}), \qquad \alpha_p(t,\mathbf{x}) := \rho(\mathbf{x})\partial_t w(t,\mathbf{x}),$$

which are respectively the *strain* and *linear momentum* and the associated Hamiltonian (total mechanical energy)

$$\mathcal{H}(t) = \frac{1}{2}\int_\Omega \boldsymbol{\alpha}_q(t,\mathbf{x})^\top\cdot\overline{\overline{T}}(\mathbf{x})\cdot\boldsymbol{\alpha}_q(t,\mathbf{x}) + \frac{1}{\rho(\mathbf{x})}\alpha_p(t,\mathbf{x})^2\, d\mathbf{x},$$

then the corresponding co-energy variables, $\boldsymbol{e} = \begin{bmatrix}\boldsymbol{e}_q^\top & e_p\end{bmatrix}$, obtained from the variational derivative of the Hamiltonian w.r.t. the energy variables, are the *stress* and *velocity*

$$\boldsymbol{e}_q := \delta_{\boldsymbol{\alpha}_q}\mathcal{H} = \overline{\overline{T}}\cdot\boldsymbol{\alpha}_q, \qquad e_p := \delta_{\alpha_p}\mathcal{H} = \frac{1}{\rho}\alpha_p\,. \tag{1}$$

Henceforth, we end up with an infinite-dimensional Hamiltonian system of the form

$$\begin{bmatrix}\partial_t\boldsymbol{\alpha}_q\\ \partial_t\alpha_p\end{bmatrix} = \begin{bmatrix}0 & \mathbf{grad}\\ \operatorname{div} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_q\\ e_p\end{bmatrix}, \tag{2}$$

$$Z\,\boldsymbol{e}_q\cdot\mathbf{n} + e_p = 0\,. \tag{3}$$

The power balance of this system is computed by the time derivative of the Hamiltonian along the trajectories, it reads thanks to *Green's formula*

$$\frac{d}{dt}\mathcal{H}(t) = -\left\|\sqrt{Z}\ \boldsymbol{e}_q\cdot\mathbf{n}\right\|^2_{L^2(\partial\Omega)} \leq 0, \tag{4}$$

where $\|\cdot\|_{L^2(\partial\Omega)}$ is the norm on $L^2(\partial\Omega)$.

More compactly, (2) can be rewritten as

$$\partial_t\boldsymbol{\alpha} = \mathcal{J}\boldsymbol{e} \quad\text{or}\quad \partial_t\boldsymbol{\alpha} = \mathcal{J}\mathcal{Q}\boldsymbol{\alpha}\,,$$

where $\mathcal{J} := \begin{bmatrix}0 & \mathbf{grad}\\ \operatorname{div} & 0\end{bmatrix}$ is the unbounded interconnection operator, and $\mathcal{Q} := \begin{bmatrix}\overline{\overline{T}} & 0\\ 0 & 1/\rho\end{bmatrix}$ is the bounded operator containing the physical parameters of the system, it is symmetric and positive definite. Finally, (3) also writes with the energy variables as: $Z\,(\overline{\overline{T}}\cdot\boldsymbol{\alpha}_q)\cdot\mathbf{n} + \frac{\alpha_p}{\rho} = 0$.

In 1D, the critical value of impedance providing the balance between Dirichlet trace of the velocity and normal trace of the stress in (3) is called *characteristic impedance* Z_c. When $Z < Z_c$ we get a dominant Dirichlet boundary condition, while for $Z > Z_c$ the normal trace of the stress is dominant. From acoustical literature, if $\overline{\overline{T}} = T_0\overline{\overline{I}}_2$ and $\rho = \rho_0$ are constant then the characteristic impedance is given by $Z_c \equiv \sqrt{T_0\rho_0}$.

2.3 Mathematical framework

Let $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega) := L^2(\Omega)^2$ the usual Hilbert spaces associated to the scalar products $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Omega$. Then $H^1(\Omega)$ and $\boldsymbol{H}^{\operatorname{div}}(\Omega)$ are the usual Sobolev spaces, where $\boldsymbol{H}^{\operatorname{div}}(\Omega) := \left\{\boldsymbol{v}_q \in \boldsymbol{L}^2(\Omega)^N, \operatorname{div}\boldsymbol{v}_q \in L^2(\Omega)\right\}$.

Theorem 1. (Kurula and Zwart (2015)).
For all initial data $(\boldsymbol{\alpha}_q^0, \alpha_p^0) \in \overline{\overline{T}}^{-1}\boldsymbol{H}^{\mathrm{div}}(\Omega)\times\rho H^1(\Omega)$, there exists a unique solution $(\boldsymbol{\alpha}_q, \alpha_p)$ satisfying (2)–(3) in

$$\mathcal{C}\big(0,t;\overline{\overline{T}}^{-1}\boldsymbol{H}^{\mathrm{div}}(\Omega)\times\rho H^1(\Omega)\big)\cap\mathcal{C}^1\big(0,t;\boldsymbol{L}^2(\Omega)\times L^2(\Omega)\big).$$

3. PARTITIONED FINITE ELEMENT METHOD (PFEM)

This method is a recent structure-preserving method for port-Hamiltonian systems, first published in Cardoso-Ribeiro et al. (2018), and more developped in Cardoso-Ribeiro et al. (2019). The principle of the method lies on the application of *Stokes' theorem* [1] only on a subset of equations that contains the control term; note that a link can be made with Farle et al. (2013), where the authors make use of exterior derivatives. The description of the application of the method to our problem is detailed below.

The proposed geometric discretization of problem (2)–(3) proceeds in four steps

(1) write the pHs of a lossless boundary controlled and observed wave equation;
(2) write this pHs in weak formulation;
(3) apply PFEM on it;
(4) use an output feedback control law to recover (3).

3.1 A lossless wave equation as pHs

Let us consider the following boundary controlled and observed wave equation

$$\begin{cases} \rho\partial_{tt}^2 w = \mathrm{div}(\overline{\overline{T}}\,\mathbf{grad}\, w),\\ u_\partial = \partial_t w\big|_{\partial\Omega},\\ y_\partial = \overline{\overline{T}}\,\mathbf{grad}\, w\cdot\mathbf{n}\big|_{\partial\Omega}. \end{cases} \tag{5}$$

Following Section 2, the associated pHs reads

$$\begin{cases} \begin{bmatrix}\partial_t\boldsymbol{\alpha}_q\\ \partial_t\alpha_p\end{bmatrix} = \begin{bmatrix}0 & \mathbf{grad}\\ \mathrm{div} & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_q\\ e_p\end{bmatrix},\\ u_\partial = e_p\big|_{\partial\Omega},\\ y_\partial = \boldsymbol{e}_q\cdot\mathbf{n}\big|_{\partial\Omega}. \end{cases} \tag{6}$$

Its power balance is then: $\frac{d}{dt}\mathcal{H}(t) = \langle u_\partial, y_\partial\rangle_{\frac12,-\frac12}$, where $\langle\cdot,\cdot\rangle_{\frac12,-\frac12}$ is the duality bracket of $H^{\frac12}(\partial\Omega)\times H^{-\frac12}(\partial\Omega)$.

3.2 Weak formulation

The dynamical part of pHs (6) in weak form reads

$$\begin{aligned}(\partial_t\boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega &= (\,\mathbf{grad}\, e_p, \boldsymbol{v}_q)_\Omega,\\ (\partial_t\alpha_p, v_p)_\Omega &= (\mathrm{div}\,\boldsymbol{e}_q, v_p)_\Omega,\end{aligned} \tag{7}$$

where $\boldsymbol{v}_q$ and v_p are sufficiently smooth test functions.

Applying *Green's formula* on the subsystem containing the control term, it appears explicitly in the new system: in the sequel, since the control term $u_\partial = e_p\big|_{\partial\Omega}$ is chosen, then the corresponding subsystem to be integrated is (7). Thus

$$\begin{aligned}(\partial_t\boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega &= -(e_p, \mathrm{div}\,\boldsymbol{v}_q)_\Omega + (\overbrace{e_p}^{u_\partial}, v_p)_{\partial\Omega},\\ (\partial_t\alpha_p, v_p)_\Omega &= (\mathrm{div}\,\boldsymbol{e}_q, v_p)_\Omega.\end{aligned}$$

[1] in our problem we use a particular case: *Green's formula*.

Note that the physical meaning of this control is not addressed here, as the aim is to provide an accurate discretization of (3).

3.3 PFEM

The weak form of (6), with $\boldsymbol{v}_q$, v_p and v_∂ sufficiently smooth scalar or vector-valued functions, is then

$$\begin{cases} (\partial_t\boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega = -(e_p, \mathrm{div}\,\boldsymbol{v}_q)_\Omega + (u_\partial, \boldsymbol{v}_q\cdot\mathbf{n})_{\partial\Omega},\\ (\partial_t\alpha_p, v_p)_\Omega = (\mathrm{div}\,\boldsymbol{e}_q, v_p)_\Omega,\\ \langle v_\partial, y_\partial\rangle_{\frac12,-\frac12} = \langle v_\partial, \boldsymbol{e}_q\cdot\mathbf{n}\rangle_{\frac12,-\frac12}. \end{cases} \tag{8}$$

A suitable approximation of (8), starts by choosing families of finite-dimensional bases (in general polynomials, conforming with the mathematical framework). Then the proposed families of approximation are

$$\begin{aligned}\mathcal{V}_q &:= \mathrm{span}\{\boldsymbol{\varphi}_q^i\}_{1\le i\le N_q},\\ \mathcal{V}_p &:= \mathrm{span}\{\varphi_p^k\}_{1\le k\le N_p},\\ \mathcal{V}_\partial &:= \mathrm{span}\{\psi_\partial^m\}_{1\le m\le N_\partial}.\end{aligned}$$

Henceforth, we define the notation

$$\begin{aligned}\boldsymbol{\alpha}_q(t,\mathbf{x}) &\approx \boldsymbol{\alpha}_q^d(t,\mathbf{x}) := \Sigma_{i=1}^{N_q}\alpha_q^i(t)\boldsymbol{\varphi}_q^i(\mathbf{x}) = \Phi_q^\top\cdot\underline{\alpha}_q,\\ \boldsymbol{e}_q(t,\mathbf{x}) &\approx \boldsymbol{e}_q^d(t,\mathbf{x}) := \Sigma_{i=1}^{N_q}e_q^i(t)\boldsymbol{\varphi}_q^i(\mathbf{x}) = \Phi_q^\top\cdot\underline{e}_q,\end{aligned} \tag{9}$$

where $\boldsymbol{\alpha}_q^i$ and $\boldsymbol{e}_q^i$ are the coefficients of $\boldsymbol{\alpha}_q^d$ and $\boldsymbol{e}_q^d$ in the basis $\mathcal{V}_q$, and $\Phi_q := \big[\boldsymbol{\varphi}_q^1, ..., \boldsymbol{\varphi}_q^{N_q}\big]^\top \in \mathbb{R}^{N_q\times 2}$, $\underline{\alpha}_q := \big[\alpha_q^1, ..., \alpha_q^{N_q}\big]^\top \in \mathbb{R}_q^N$ and $\underline{e}_q := \big[e_q^1, ..., e_q^{N_q}\big]^\top \in \mathbb{R}_q^N$.
Likewise for the rest of the variables, we define

$$\begin{aligned}\alpha_p(t,\mathbf{x}) &\approx \alpha_p^d(t,\mathbf{x}) := \Sigma_{k=1}^{N_p}\alpha_p^k(t)\varphi_p^k(\mathbf{x}) = \Phi_p^\top\cdot\underline{\alpha}_p,\\ e_p(t,\mathbf{x}) &\approx e_p^d(t,\mathbf{x}) := \Sigma_{k=1}^{N_p}e_p^k(t)\varphi_p^k(\mathbf{x}) = \Phi_p^\top\cdot\underline{e}_p,\end{aligned} \tag{10}$$

where $\Phi_p := \big[\varphi_p^1, ..., \varphi_p^{N_p}\big]^\top \in \mathbb{R}^{N_p\times 1}$.
Similarly for the boundary variables

$$\begin{aligned}u_\partial(t,\mathbf{x}) &\approx u_\partial^d(t,\mathbf{x}) := \Sigma_{m=1}^{N_\partial}u_\partial^k(t)\psi_\partial^m(\mathbf{x}) = \Psi_\partial^\top\cdot\underline{u}_\partial,\\ y_\partial(t,\mathbf{x}) &\approx y_\partial^d(t,\mathbf{x}) := \Sigma_{m=1}^{N_\partial}y_\partial^k(t)\psi_\partial^m(\mathbf{x}) = \Psi_\partial^\top\cdot\underline{y}_\partial\end{aligned} \tag{11}$$

where $\Psi_\partial := \big[\psi_\partial^1, ..., \psi_\partial^{N_\partial}\big]^\top \in \mathbb{R}^{N_\partial\times 1}$.
Plugging the approximation (9)–(10)–(11) into the weak form (8), and choosing any test functions $\boldsymbol{v}_q$, v_p and v_∂ in $\mathcal{V}_q$, $\mathcal{V}_p$, and $\mathcal{V}_\partial$ leads to

$$\begin{cases} \Sigma_{i=1}^{N_q}\big(\boldsymbol{\varphi}_q^i, \boldsymbol{\varphi}_q^j\big)_\Omega\dfrac{d}{dt}\alpha_q^i = -\Sigma_{k=1}^{N_p}\big(\varphi_p^k, \mathrm{div}\,\boldsymbol{\varphi}_q^j\big)_\Omega e_p^k,\\ \qquad\qquad + \big(\Sigma_{m=1}^{N_\partial}\psi_\partial^m, \boldsymbol{\varphi}_q^j\cdot\mathbf{n}\big)_{\partial\Omega}u_\partial^k,\\ \qquad\qquad\qquad\qquad j = 1, ..., N_q\\ \Sigma_{k=1}^{N_p}\big(\varphi_p^k, \varphi_p^\ell\big)_\Omega\dfrac{d}{dt}\alpha_p^k = \Sigma_{i=1}^{N_q}\big(\mathrm{div}\,\boldsymbol{\varphi}_q^i, \varphi_p^\ell\big)_\Omega e_q^i, \ell = 1, ..., N_p\\ \big(\Sigma_{m=1}^{N_\partial}\psi_\partial^m, \psi_\partial^n\big)_{\partial\Omega}y_\partial^k = \big(\Sigma_{i=1}^{N_q}\boldsymbol{\varphi}_q^i\cdot\mathbf{n}, \psi_\partial^n\big)_{\partial\Omega}e_q^i, n = 1, .., N_\partial \end{cases}$$

In matrix form, we obtain

$$\begin{cases} M_q\ \dfrac{d}{dt}\underline{\alpha}_q = D\ \underline{e}_p + B\underline{u}_\partial\\ M_p\ \dfrac{d}{dt}\underline{\alpha}_p = -D^\top\ \underline{e}_q\\ \qquad M_\partial\underline{y}_\partial = B^\top\underline{e}_q \end{cases} \tag{12}$$

where

$$
\begin{aligned}
\big(M_q\big)_{ij} &= \big(\boldsymbol{\varphi}_q^j, \boldsymbol{\varphi}_q^i\big)_\Omega, & M_q &= \int_\Omega \Phi_q \cdot \Phi_q^\top \in \mathbb{R}^{N_q \times N_q},\\
\big(M_p\big)_{k\ell} &= \big(\varphi_p^\ell, \varphi_p^k\big)_\Omega, & M_p &= \int_\Omega \Phi_p \cdot \Phi_q^\top \in \mathbb{R}^{N_p \times N_p},\\
\big(D\big)_{jk} &= -\big(\varphi_p^k, \operatorname{div} \boldsymbol{\varphi}_q^j\big)_\Omega, & D &= -\int_\Omega \operatorname{div} \Phi_q \cdot \Phi_p^\top \in \mathbb{R}^{N_q \times N_p},\\
\big(B\big)_{jm} &= \big(\psi_\partial^m, \boldsymbol{\varphi}_q^j \cdot \mathbf{n}\big)_{\partial\Omega}, & B &= \int_{\partial\Omega} \Phi_q \cdot \mathbf{n} \cdot \Psi_\partial^\top \in \mathbb{R}^{N_q \times N_\partial},\\
\big(M_\partial\big)_{mn} &= \big(\psi_\partial^n, \psi_\partial^m\big)_{\partial\Omega}, & M_\partial &= \int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top \in \mathbb{R}^{N_\partial \times N_\partial}.
\end{aligned}
$$

System (12) is a finite-dimensional pHs, as defined in Egger et al. (2018), i.e. including mass matrices.

For simulation and control purposes, it is very interesting to get the discrete counterpart of $\mathcal{Q}$ and substitute the discrete part of the variable $\boldsymbol{e}$ by $\boldsymbol{\alpha}$. To do so, we rewrite the constitutive relation (1) in a weak sense

$$(\boldsymbol{e}_q, \boldsymbol{v}_q)_\Omega = (\overline{\overline{T}}\boldsymbol{\alpha}_q, \boldsymbol{v}_q)_\Omega$$

by writing and testing $\boldsymbol{e}_q$ and $\boldsymbol{\alpha}_q$ over the family of basis functions $\{\boldsymbol{\varphi}_q^i\}_{1\le i\le N_q}$, we have

$$\Big[\int_\Omega \Phi_q \cdot \Phi_q^\top\Big]\, \underline{e}_q = \Big[\int_\Omega \Phi_q \cdot \overline{\overline{T}}\, \Phi_q^\top\Big]\, \underline{\alpha}_q\,,$$

then we can define Q_T as the spatially averaged value of $\overline{\overline{T}}$ on the basis functions $\{\boldsymbol{\varphi}_q^i\}_{1\le i\le N_q}$ as

$$
\begin{aligned}
\underline{e}_q &= \underbrace{\Big[\int_\Omega \Phi_q \cdot \Phi_q^\top\Big]^{-1}\Big[\int_\Omega \Phi_q \cdot \overline{\overline{T}}\, \Phi_q^\top\Big]}_{Q_T}\, \underline{\alpha}_q\,,\\
\implies \quad \underline{e}_q &= Q_T\, \underline{\alpha}_q, \qquad Q_T \in \mathbb{R}^{N_q \times N_q}\,.
\end{aligned}
$$

For example in a homogeneous isotropic problem, *i.e.* $\overline{\overline{T}} = T_0 \overline{\overline{I}}_2$ with $T_0 \equiv cst$, then $Q_T = T_0 I_{N_q}$, *i.e.* $\underline{e}_q = T_0 \underline{\alpha}_q$. Likewise for $e_p = \frac{1}{\rho}\alpha_p$, we define Q_ρ as

$$
\begin{aligned}
\underline{e}_p &= \underbrace{\Big[\int_\Omega \Phi_p \cdot \Phi_p^\top\Big]^{-1}\Big[\int_\Omega \Phi_p \cdot \frac{1}{\rho}\, \Phi_p^\top\Big]}_{Q_\rho}\, \underline{\alpha}_p\\
\implies \quad \underline{e}_p &= Q_\rho\, \underline{\alpha}_p, \qquad Q_\rho \in \mathbb{R}^{N_p \times N_p}.
\end{aligned}
$$

Hence, we can write (12) as

$$
\begin{cases}
\begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix} \dfrac{d}{dt} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix} = \begin{bmatrix} 0 & D \\ -D^\top & 0 \end{bmatrix} \begin{bmatrix} Q_T & 0 \\ 0 & Q_\rho \end{bmatrix} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \underline{u}_\partial,\\
\underline{y}_\partial = M_\partial^{-1}\, B^\top\, Q_T\, \underline{\alpha}_q.
\end{cases}
\tag{13}
$$

Since the finite element approach discretizes identity into mass matrix, we define the new scalar products associated to these matrices

$$
\begin{aligned}
\big\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \big\rangle_q &:= \boldsymbol{v}_1^\top\, M_q\, \boldsymbol{v}_2, \quad \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^{N_q},\\
\big\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \big\rangle_p &:= \boldsymbol{v}_1^\top\, M_p\, \boldsymbol{v}_2, \quad \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^{N_p},\\
\big\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \big\rangle_\partial &:= \boldsymbol{v}_1^\top\, M_\partial\, \boldsymbol{v}_2, \quad \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^{N_\partial},
\end{aligned}
$$

where $\big\langle \cdot, \cdot \big\rangle_\partial$ is the scalar product associated to the mass matrix M_∂, not to be confused with the duality bracket.

Consequently, the matrices Q_T and Q_ρ are symmetric positive definite with respect to the scalar products $\big\langle \cdot, \cdot \big\rangle_q$ and $\big\langle \cdot, \cdot \big\rangle_p$. Also y_∂ is exactly the conjugate output of u_∂ with respect to $\big\langle \cdot, \cdot \big\rangle_\partial$.

Discrete Hamiltonian Then we define the discrete Hamiltonian as

$$
\begin{aligned}
\mathcal{H}_d(t) &:= \frac{1}{2}\int_\Omega \boldsymbol{\alpha}_q^d \cdot \overline{\overline{T}} \cdot \boldsymbol{\alpha}_q^d + \alpha_p^d \frac{1}{\rho} \alpha_p^d\\
&= \frac{1}{2}\Big(\underline{\alpha}_q^\top \Big[\int_\Omega \Phi_q \cdot \overline{\overline{T}} \cdot \Phi_q^\top\Big]\, \underline{\alpha}_q + \underline{\alpha}_p^\top \Big[\int_\Omega \Phi_p\, \frac{1}{\rho}\, \Phi_p^\top\Big]\, \underline{\alpha}_p\Big)\\
&= \frac{1}{2}\Big(\underline{\alpha}_q^\top\, M_q\, Q_T\, \underline{\alpha}_q \;+\; \underline{\alpha}_q^T\, M_q\, Q_\rho\, \underline{\alpha}_p\Big)\\
&= \frac{1}{2}\big\langle \underline{\alpha}_q, Q_T\, \underline{\alpha}_q \big\rangle_q + \frac{1}{2}\big\langle \underline{\alpha}_p, Q_\rho\, \underline{\alpha}_p \big\rangle_p.
\end{aligned}
$$

Moreover, the discrete power balance is

$$\frac{d}{dt}\mathcal{H}_d(t) = \underline{y}_\partial^\top M_\partial \underline{u}_\partial := \big\langle u_\partial, y_\partial \big\rangle_\partial, \tag{14}$$

which perfectly fits with the pHs framework of Egger et al. (2018).

3.4 Closed-Loop system

The boundary condition (3), $Z\, \boldsymbol{e}_q \cdot \mathbf{n} + e_p = 0$, corresponds to an output feedback law $-Z$, hence system (6) can be rewritten with $u_\partial = -Z y_\partial$, and once again, the closure relation is to be approximated in a weak sense

$$
\begin{aligned}
\big(u_\partial, \psi_\partial\big)_{\partial\Omega} &= -\big(Z y_\partial, \psi_\partial\big)_{\partial\Omega},\\
\Big[\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top\Big]\, \underline{u}_\partial &= -\Big[\int_{\partial\Omega} \Psi_\partial \cdot Z\, \Psi_\partial^\top\Big]\, \underline{y}_\partial,
\end{aligned}
$$

then we can define Z_d as

$$
\begin{aligned}
\underline{u}_\partial &= -\underbrace{\overbrace{\Big[\int_{\partial\Omega} \Psi_\partial \cdot \Psi_\partial^\top\Big]^{-1}}^{M_\partial^{-1}}\overbrace{\Big[\int_{\partial\Omega} \Psi_\partial \cdot Z\, \Psi_\partial^\top\Big]}^{M_Z}}_{Z_d}\, \underline{y}_\partial,\\
\implies \quad \underline{u}_\partial &= -Z_d \underline{y}_\partial, \quad Z_d = M_\partial^{-1}\, M_Z, \quad Z_d, M_z \in \mathbb{R}^{N_\partial \times N_\partial}.
\end{aligned}
$$

Hence

$$\underline{u}_\partial = -Z_d\, M_\partial^{-1}\, B^\top\, Q_T\, \underline{\alpha}_q.$$

Thus we can plug the latter expression in the discrete system (13)

$$
\begin{cases}
M_q \dfrac{d}{dt}\underline{\alpha}_q = D\; Q_\rho\; \underline{\alpha}_p - B\, Z_d\, M_\partial^{-1}\, B^\top\, Q_T\, \underline{\alpha}_q\\
M_p \dfrac{d}{dt}\underline{\alpha}_p = -D^\top Q_T\; \underline{\alpha}_q
\end{cases}
$$

Let us set

$$R_Z := B\; Z_d\; M_\partial^{-1} B^\top, \tag{15}$$

then the finite-dimensional dissipative Hamiltonian system derived from system (2)–(3) is

$$\begin{bmatrix} M_q & 0 \\ 0 & M_p \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix} = \begin{bmatrix} -R_Z & D \\ -D^\top & 0 \end{bmatrix} \begin{bmatrix} Q_T & 0 \\ 0 & Q_\rho \end{bmatrix} \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix}.$$

More compactly

$$M_d \frac{d}{dt}\underline{\alpha} = (J_d - R)\; Q_d\; \underline{\alpha}, \tag{16}$$

where $\underline{\alpha} = \begin{bmatrix} \underline{\alpha}_q \\ \underline{\alpha}_p \end{bmatrix}$, $M_d = \begin{bmatrix} M_q & 0 \\ 0 & M_q \end{bmatrix}$, $J_d = \begin{bmatrix} 0 & D \\ -D^\top & 0 \end{bmatrix}$, $Q_d = \begin{bmatrix} Q_T & 0 \\ 0 & Q_\rho \end{bmatrix}$ and $R = \begin{bmatrix} R_Z & 0 \\ 0 & 0 \end{bmatrix} \geq 0$, furthermore R_Z is of maximum rank N_∂, since M_Z is.

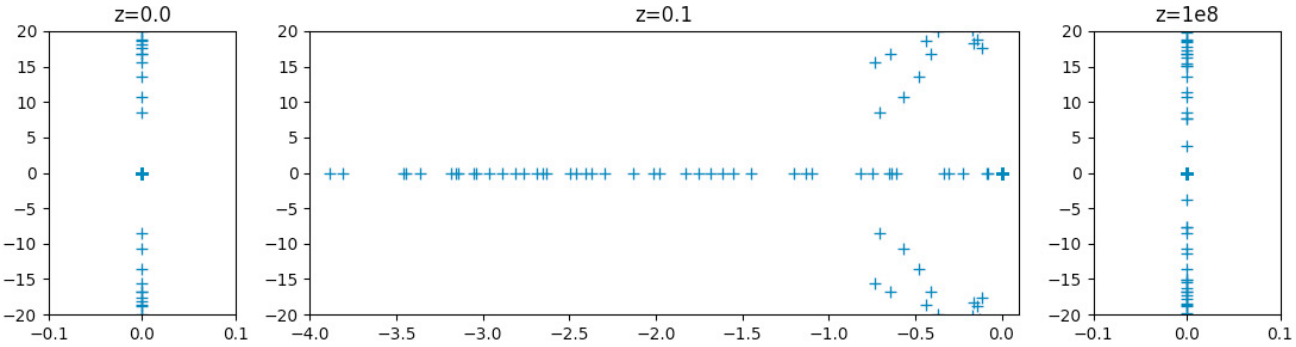

Figure 1. The spectrum for $L_x = 2$, $\overline{\overline{T}} = \overline{\overline{I}}_2$ and $\rho = \frac{1}{6}$ for: (left): $Z = 0$; (center): $Z = 0.1$; (right): $Z = 10^8 \approx \infty$.

Discrete Hamiltonian We get the discrete power balance, counterpart of (4), from (14)

$$\frac{d}{dt}\mathcal{H}_d(t) = \langle u_\partial, y_\partial \rangle_\partial = -\langle Z_d\, y_\partial, y_\partial \rangle_\partial \leq 0. \tag{17}$$

Indeed, Z_d is a positive semi-definite matrix with respect to $\langle \cdot, \cdot \rangle_\partial$, since $\langle Z_d\, y_\partial, y_\partial \rangle_\partial := y_\partial^\top M_\partial Z_d\, y_\partial = y_\partial^\top M_Z\, y_\partial$ and M_Z is positive semi-definite by construction.

4. NUMERICAL SIMULATION: TWO CASE STUDIES

In § 4.1, the classical test case of a uniform impedance is investigated, and an optimal value of this impedance is discussed. Then, in § 4.2, our method is applied to a more complex test case, providing coherent results.

The simulation and computation below are performed using FEniCS software and Python 3. We consider a rectangle domain $\Omega = (0, L_x) \times (0, 1)$ and the chosen finite elements families are RT_0 (lowest order Raviart-Thomas element), P_1 (Lagrange element) and P_1 respectively for q, p and ∂ variables. The associated numbers of degrees of freedom are $N_q = 369$, $N_p = 140$ and $N_\partial = 48$.

4.1 A simple test case: isotropic, homogeneous medium with uniform impedance at the boundary

Following Kergomard et al. (2006) in the 1D case, where both the spectrum and the eigenfunctions are analytically computed (the Riesz basis property of the eigenfunctions is also fully proved), we can infer some spectral properties of the 2D case when Ω is rectangle.

First, we look at the spectrum for three different values of $Z = 0, 0.1, \infty$. As said in § 2.1, $Z = 0$ and $Z = \infty$ mean that the system is conservative, implying that their spectrums must lie on $i\mathbb{R}$, which is confirmed in the left and right plot in Figure 1 (numerically we chose $\infty \approx 10^8$). For $Z = 0.1$, we see that the eigenvalues of $Z = 0$ moved from $i\mathbb{R}$ to the left half-plane.

Remark 1. Since zero is always an eigenvalue, one may show that the proposed method preserves also the stationary fields.

Optimal value of the impedance It is interesting to see the behavior of the spectrum in term of Z. Then, Figure 2 shows the largest non-null real part of the eigenvalues of the system (16) (*i.e.* $-\max_{\lambda_n \neq 0} \Re e(\lambda_n)$) as a function of Z. The computations are done for $T_0 = 5$ and $\rho_0 = 3$, then the peak in Figure 2 allows to find the numerical optimal impedance $Z_{opt} = 1.16$ in order to obtain the optimal decay rate in (17).

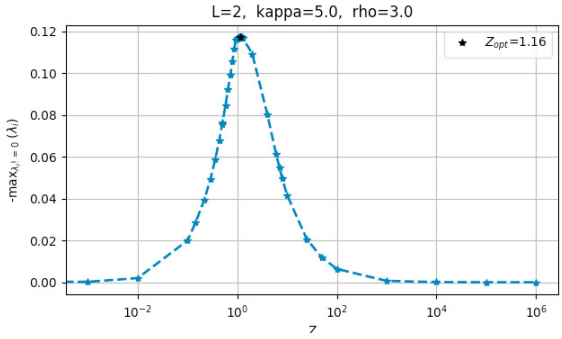

Figure 2. Decay rate of $\mathcal{H}$ as a function of Z

Another important point already emphasized on Figure 1 is remarkable on Figure 2: if $Z = 0, \infty$, we indeed have a conservative equation, *i.e.* $\mathcal{J}\mathcal{Q}$ is skew-adjoint in the appropriate functional space.

4.2 A more involved test case: anisotropic, heterogeneous medium with space-varying impedance at the boundary

For an anisotropic heterogeneous case, we compute the discrete Hamiltonian (Figure 3), with physical parameters ρ, $\overline{\overline{T}}$ and Z taking different values in the domain and at the boundary. Let $L_x = 3$, $\Omega = \bigcup_{i=1}^4 \Omega_i$ and $\partial\Omega = \bigcup_{i=1}^4 \Gamma_i$, we set $\overline{\overline{T}}|_{\Omega_1 \cup \Omega_3} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, $\overline{\overline{T}}|_{\Omega_2 \cup \Omega_4} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\rho|_{\Omega_1 \cup \Omega_3} = 0.5$, $\rho|_{\Omega_2 \cup \Omega_4} = 1$, $Z|_{\Gamma_1 \cup \Gamma_3} = 1$ and $Z|_{\Gamma_2 \cup \Gamma_4} = 0.5$. Figure 3 shows the evolution of the discrete Hamiltonian over time. In $t \in [0, \frac{1}{3}t_f]$, we have a conservative system: we do not apply boundary control, indeed we see the conservation of the Hamiltonian. Just after, for $t > \frac{1}{3}t_f$, we apply the output feedback control law, mimicking the impedance boundary condition, then the system becomes dissipative. Note however that it does not decay to 0 due to the existence of stationary solutions ($\lambda = 0$ belongs to the spectrum).

The time integration is performed thanks to Crank-Nicholson scheme (2nd order accurate, unconditionally stable) with $t_f = 20$ and $dt = 10^{-3}$. The initial data are a trigonometric vector function for $\boldsymbol{\alpha}_q(0)$ and zero for $\alpha_p(0)$, which prove compatible with the boundary conditions.

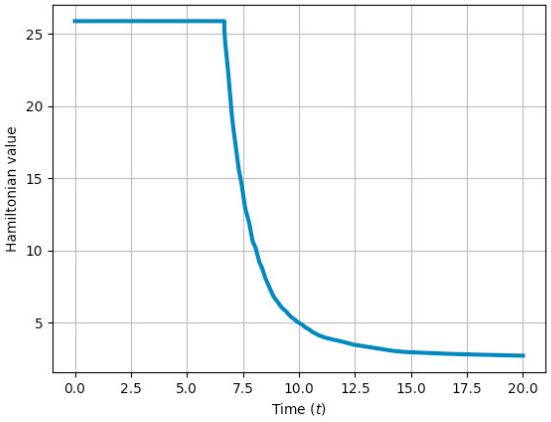

Figure 3. Decreasing Hamiltonian for the anisotropic heterogeneous medium with space-varying impedance.

5. CONCLUSION AND PERSPECTIVES

To sum up, a structure-preserving discretization of the anistropic heterogeneous wave equation with boundary damping has been presented. From the proposed strategy, the dissipation matrix R that was hidden in $D(\mathcal{J}\mathcal{Q})$ in the continuous model has been derived, and, thanks to an equivalent formulation in terms of output feedback control law, a meaningful factorization of R that makes its low rank obvious has been obtained. New scalar products associated to the mass matrices fit perfectly into the finite element framework, furthermore they allow for recovering a discrete Hamiltonian system from the discrete Hamiltonian $\mathcal{H}_d$. The overall theoretical and numerical work has been illustrated by simulation results, both on the well-known constant coefficient cases and on some more general situation, either anisotropic, heterogeneous or with space-varying impedance.

As interesting perspectives in the near future, let us mention first the optimal order of convergence of PFEM depending on the choice of the finite elements, see Serhani et al. (2019), and second the impact of the refinement of the mesh at the boundary on the decay rate of the damped solutions.
Then, in the spirit of Egger et al. (2018), a structure-preserving reduction procedure could eventually be applied to such a dissipative system in order to reduce it to a relatively small dimension for efficient simulation and control purposes.
Finally, in many applications, e.g. in acoustics, the problem involves a time-varying impedance and gives rise to dynamic boundary damping; the associated filters are either of low-pass type $Z(s) = \frac{Z_0}{1+\tau s}$, or of high-pass type $Z(s) = Z_\infty \frac{\tau s}{1+\tau s}$, and even non-rational, see e.g. Monteghetti et al. (2018): applying PFEM to these systems would be a major breakthrough for aeronautical applications.

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