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Stability of Linear Fractional Differential Equations with Delays: a coupled Parabolic-Hyperbolic PDEs formulation. ⋆

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Abstract: Fractional differential equations with delays are ubiquitous in physical systems, a recent example being time-domain impedance boundary conditions in aeroacoustics. This work focuses on the derivation of delay-independent stability conditions by relying on infinite-dimensional realisations of both the delay (transport equation, hyperbolic) and the fractional derivative (diffusive representation, parabolic). The stability of the coupled parabolic-hyperbolic PDE is studied using straightforward energy methods. The main result applies to the vector-valued case. As a numerical illustration, an eigenvalue approach to the stability of fractional delay systems is presented.

Keywords: Systems with time-delays, stability of delay systems, fractional-order systems, Lyapunov methods, semigroup and operator theory, diffusive representation.

1. INTRODUCTION

Time-delay and fractional operators are ubiquitous in physics. The former model lossless convection phenomena (think wave equation), while the latter model "viscous" losses (i.e. lossy mechanisms that exhibit a long-memory effect, think heat equation). As a result, they have been studied by many different communities; below is a crude overview of the literature, biased towards stability results.

The theory of time-delay systems has been enjoying a sustained development since the 1960s: let us only cite the classic monograph of Bellman and Cooke (1963). Stability criteria roughly split into two: frequency-domain and time-domain ones. Frequency-domain approaches rely on studying the characteristic equation to locate (or merely count) unstable poles (Michiels and Niculescu, 2014, Chap. 1). In the time domain, a popular method consists in designing a Lyapunov-Krasovskii functional and approximating the corresponding sufficient stability condition as a linear matrix inequality (LMI) that is tractable numerically: see (Fridman, 2014, Chap. 3) and (Briat, 2014, Chap. 5) for an introduction; Seuret et al. (2015) and Baudouin et al. (2016) for advanced LMIs of arbitrary accuracy.
Both visions are unified by remarking that time-delay operators can be realised (in the sense of systems theory) as transport equations (of *hyperbolic* nature). This fact is commonly used by the time-delay community as a means of computing zeros as eigenvalues (Michiels and Niculescu, 2014, § 2.2), as well as by the PDE community to show well-posedness of the corresponding infinite-dimensional Cauchy problem (Engel and Nagel, 2000, § VI.6), (Curtain and Zwart, 1995, § 2.4).

The theory of fractional calculus is also well-established. The stability of fractional systems of commensurate order can be fully characterised algebraically: no poles in the sector $|\arg(s)| \leq \alpha\,\frac{\pi}{2}$, see Matignon (1998). Moreover, fractional operators turn out to be convolution operators with a so-called "diffusive" kernel, a fact that enables to recast them into the output of an infinite-dimensional ODE (of *parabolic* nature), related to the heat equation. See Matignon (2009) for an overview chapter on both aspects. Applications of this parabolic realisation include, but are not limited to: numerical simulation (Lombard and Matignon (2016)), well-posedness and stability of fractional ODE (Matignon and Prieur (2005)) and PDE (Matignon and Prieur (2014)).

The theory of fractional delay differential equations is more recent and limited. A necessary and sufficient stability condition in the frequency domain is established in Bonnet and Partington (2002) for the commensurate delay case. (A sufficient condition for the non-commensurate case is proven in Deng et al. (2007), provided that the delay and fractional operators are not composed.) A corresponding numerical method and its MATLAB implementation are available, see Fioravanti et al. (2012) and Avanessoff et al. (2013). More recently, a frequency-sweeping approach that counts unstable poles has been proposed for systems of retarded type, see Zhang et al. (2016).

This work focuses on the derivation of delay-independent stability results for linear fractional differential equations with delays, making use of the language of systems theory and PDEs. The method of proof is based on realisations in

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the sense of (infinite-dimensional) systems theory: *hyperbolic* realisation for the time-delay and *parabolic* realisation for the fractional derivative.
This article is organised as follows. Sec. 2 presents a motivation for this work. Sec. 3 works out a scalar toy model to introduce the method of proof, and contrasts it with a frequency-domain method. Sec. 4 proves the main theoretical result of this paper, namely Thm. 7. Sec. 5 proposes an eigenvalue approach to study the stability of fractional delay systems, and provides numerical illustrations. Sec. 6 covers possible extensions and open questions.

2. AN APPLICATION IN AEROACOUSTICS

The purpose of this section is to present an application of fractional delay systems in aeroacoustics. Additional material, including numerical simulations that rely on the hyperbolic-parabolic realisation, will be presented at the conference.
Context. In order to reduce the noise emitted by jet engines, their inlets are lined with a sound absorbing material (known as a *liner*). The most widespread material is the so-called "single degree of freedom" (SDoF) liner, depicted in Fig. 1. Based on the Helmholtz resonator, it consists of a perforated plate and a hard-backed cavity. Physically, both components have a different purpose: the (short and narrow) perforations, through visco-thermal losses, account for the majority of sound absorption; the (long and large) cavities drive the resonant frequencies, through a lossy back-and-forth propagation.

Boundary condition. Acoustic liners are typically modelled as single-input single-output systems: $\hat{p} = \hat{z}\hat{u}_n$, where $\hat{p}$, $\hat{u}_n$, and $\hat{z}$ respectively denote the Laplace transforms of acoustic pressure, (inward) normal velocity, and impedance (Allard and Atalla (2009)). For the SDoF liner depicted in Fig. 1, a broadly-applicable model reads (Monteghetti et al. (2016))

$$\begin{aligned}\hat{z}(s) = {} & \overbrace{a_0 + a_\alpha\, s\, \hat{h}_\alpha(s) + a_1\, s}^{\text{perforated plate}} \\ & + \underbrace{\coth(b_0 + b_\alpha\, s\, \hat{h}_\alpha(s) + b_1\, s)}_{\text{hard-backed cavity}},\end{aligned} \tag{1}$$

where $\hat{h}_\alpha := s \mapsto s^{-\alpha}$, with $\alpha = 1/2$. The coefficient a_0 (b_0) models frequency-independent losses in the perforation (cavity), while a_α (b_α) models frequency-dependent losses. A study of (1), based on complex analysis, leads to the following causal fractional delay differential equation in the time domain

$$p(t) = a_0\, u_n(t) + a_1\, \dot{u}_n(t) + a_\alpha\, d_C^\alpha u_n(t) + D_2(u_n)(t-\tau), \tag{2}$$

where d_C^α stands for the Caputo fractional derivative (see App. B), and D_2 is a convolution operator with a diffusive

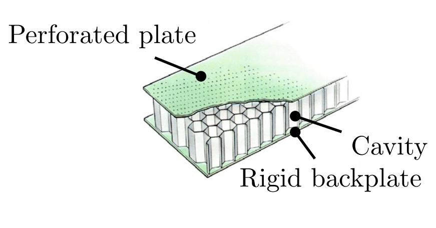

Fig. 1. Example of SDoF liner: perforated plate and hard-backed cavity.

kernel, i.e. that admits the following representation in the time domain

$$D_2(u_n)(t) = \left[\sum_{n\in\mathbb{Z}} r_n e^{s_n t} + \int_0^\infty e^{-\xi t}\, \mathrm{d}\mu(\xi)\right] \star u_n(t),$$

where the residues r_n, poles s_n and diffusive weight μ are linked to b_0, b_α and b_1. More generally, (2) can model a wide range of material, not limited to (1), provided that the fractional derivative d_C^α is replaced by a suitable diffusive operator, D_1, expressed similarly to D_2.

PDE. To model sound absorption in a duct, (2) is used as a (time-domain impedance) *boundary condition* of a PDE on (p, u) such as the linearised homentropic Euler equations (u_0 and $\gamma > 1$ denote the base flow velocity profile and specific heat ratio, respectively)

$$\begin{cases}\partial_t p + \nabla\cdot u + u_0\cdot\nabla p + \gamma\, p\nabla\cdot u_0 = 0\\ \partial_t u + \nabla p + [u_0\cdot\nabla]u + [u\cdot\nabla]u_0 + p[u_0\cdot\nabla]u_0 = 0.\end{cases} \tag{3}$$

A spatial discretisation of (2)–(3) then leads to

$$M\,\dot{X} + K\,X = F_1 d_C^\alpha(C\cdot X) + F_2\, D_2(C\cdot X)(\cdot - \tau), \tag{4}$$

where $X \in \mathbb{R}^n$ denotes the DoF of the spatial discretisation and $C\cdot X$ the DoF that belong to the impedance boundary (a subset of X).
In summary, an aeroacoustical problem (sound absorption in a duct) leads to the fractional delay differential equation (4).

3. A TOY MODEL, AS WORKED-OUT EXAMPLE

The purpose of this section is to derive sufficient delay-independent stability condition for the following scalar model ($0 < \alpha < 1$)

$$\dot{x}(t) = ax(t) + bx(t-\tau) - g d_C^\alpha x(t) \quad \text{for } t > \tau \tag{5}$$

$$x(t) := x^0(t) \quad \text{for } 0 \le t \le \tau\,. \tag{6}$$

To derive general results, we consider complex parameters $(a, b, g) \in \mathbb{C}^3$. The interest of this case is twofold: firstly, it acts as a stepping stone to the more general result given in Thm. 7; secondly, it enables to contrast the frequency-domain and time-domain (energy) methods.

Remark 1. Due to the delay τ in the evolution equation (5), the given initial datum is a *function* on the time interval $[0, \tau]$. The choice of $[0, \tau]$, which differs from the traditional convention for time-delay systems, is justified by the fact that fractional derivatives (be they Caputo or Riemann-Liouville) are naturally defined and related to one another within a *causal* setting, which implies zero values before $t = 0$.

3.1 Analysis in the Laplace domain

Let us write $g = |g|\exp(i\,\theta_g)$, and denote $\overline{\alpha} := 1 - \alpha$.

Theorem 2. Under the following algebraic condition

$$\Re(a) < -|b| \le 0\,, \tag{7}$$

and $\theta_g \in J_\alpha := [-\overline{\alpha}\,\frac{\pi}{2}, \overline{\alpha}\,\frac{\pi}{2}]$, system (5)–(6) is delay-independent stable.

Remark 3. Let us first inspect the two limiting cases for J_α: when $\alpha = 1$, $J_1 = \{0\}$ only, and the term $(1 + |g|)$ appears in front of $\dot{x}$ in the left-hand side, giving rise to a delay-independent stable delay equation without any fractional derivative; whereas when $\alpha = 0$, $J_0 = [-\frac{\pi}{2}, \frac{\pi}{2}]$, and the term $(a-g)$ appears in front of x in the right-hand side: since $\Re(-g) = -|g|\cos(\theta_g) \leq 0$, delay-independent stability holds.

Proof. For $c \in \mathbb{R}$, let us define the Laplace transform $\mathcal{L}_c(f) := \int_c^\infty f(t)\, e^{-st}\, \mathrm{d}t$, and denote $\hat{f} := \mathcal{L}_0(f)$. Firstly, take the Laplace transform of (5), taking care of the fact that it is defined for $t > \tau$: $\mathcal{L}_\tau((5))$. Secondly, simplify $\mathcal{L}_\tau(\dot{x})$, $\mathcal{L}_\tau(x(\cdot - \tau))$ and $\mathcal{L}_\tau(d_C^\alpha x)$ to recover terms that uses only the Laplace transform $\mathcal{L}_0$. Eventually, the Laplace transform of (5)–(6) reads

$$\begin{aligned}\hat{x}(s) = &\underbrace{(s-a)\hat{h}(s)\,\widehat{x^0}(s)}_{\text{(a)}} + g\underbrace{\hat{h}(s)\,\mathcal{L}_0[d_C^\alpha x^0\, \mathbb{1}_{[0,\tau]}](s)}_{\text{(b)}} \\ &+ g x^0(0)\underbrace{\hat{h}(s)\, s^\alpha}_{\text{(c)}} + g x^0(\tau)\underbrace{\hat{h}(s) s^\alpha e^{-s\tau}}_{\text{(d)}} \\ &+ x^0(\tau)\underbrace{\hat{h}(s) e^{-s\tau}}_{\text{(e)}},\end{aligned}$$

where $\hat{h}(s) := (s - a - b\exp(-\tau s) + g s^\alpha)^{-1}$ and $\mathbb{1}$ is the characteristic function. The function $\hat{h}$ is defined in the open right half-plane $\mathbb{C}_\beta^+$ for some real abscissa β; since the fractional power of s requires a cut to be performed from the branching point $s = 0$, necessarily $\beta \geq 0$. Let us check its analyticity. Let $s = x + i\,y = r\exp(i\theta)$ with $\{x \geq 0$ and $y \in \mathbb{R}\}$ or $\{r \geq 0$ and $|\theta| \leq \frac{\pi}{2}\}$. We have

$$\begin{aligned}&\Re(s - a - b\,e^{-\tau s} + g\,s^\alpha) \\ &= x - \Re(a) - |b|\exp(-\tau x)\cos(\tau y - \theta_b) + \Re(g\,s^\alpha) \\ &\geq x - \Re(a) - |b| + |g|\,|s|^\alpha \cos(\theta_g + \alpha\theta) \\ &\geq x - \Re(a) - |b| \\ &\geq -\Re(a) - |b|,\end{aligned}$$

where we used the assumption $\theta_g \in J_\alpha := [-\overline{\alpha}\frac{\pi}{2}, \overline{\alpha}\frac{\pi}{2}]$, which yields $\cos(\theta_g + \alpha\theta) \geq 0$ for any $|\theta| \leq \frac{\pi}{2}$. If additionally $\Re(a) < -|b|$ then $\hat{h}$ is analytic in the closed right half-plane $\overline{\mathbb{C}_0^+}$.

Let us now investigate the asymptotic behaviour of x for $t \to \infty$. The limit of the terms (c), (d) and (e) can be studied using the final value theorem, which yields a null value. The terms (a) and (b) require a more involved treatment. In the time domain, (b) reads $h \star (d_C^\alpha x^0\, \mathbb{1}_{[0,\tau]})$, where $h \in \mathcal{A}(0)$ (see § B.2) and $d_C^\alpha x^0\, \mathbb{1}_{[0,\tau]}$ is compactly supported in $[0, \tau]$. Using the decomposition (B.2) and the Lebesgue dominated convergence theorem enables to conclude that $(h \star (d_C^\alpha x^0\, \mathbb{1}_{[0,\tau]}))(t) \to 0$. The term (a) yields $p \star x^0$, where $p \in \mathcal{A}(0)$ and x^0 is compactly supported in $[0, \tau]$. Therefore the exact same arguments enable to conclude that $(p \star x^0)(t) \to 0$. □

3.2 Energy analysis with parabolic-hyperbolic realisation

In this section, we use the parabolic-hyperbolic realisation to establish the following result.

Proposition 4. Let $g > 0$. Under the algebraic condition

$$\Re(a) < -|b| \leq 0,$$

the system (5)–(6) with $x^0(0) = 0$ is delay-independent stable.

Remark 5. Here, the parabolic-hyperbolic realisation leads to a less general result than Thm. 2. A better result might be achieved by using less stringent estimates; however, this proof is presented only as a step towards the vector-valued case (see the conditions on G in Thm. 7).

Proof. We begin by using (B.1) to recast (5) using only the Riemann-Liouville fractional derivative

$$\dot{x}(t) = a\,x(t) + b\,x(t-\tau) - g\,D_{\mathrm{RL}}^\alpha\, x(t) + g\,w(t), \quad (8)$$

where $w(t) = x^0(0)\,h_{1-\alpha}(t) = 0$. For the sake of compactness, let us denote $x_\tau(\cdot) := x(\cdot - \tau)$. The natural energy functional is $E_x := \frac{1}{2}|x|^2$, and its decay rate along the trajectories is given by

$$\dot{E}_x = 2\,\Re(a)\,E_x + \Re(\overline{x}\,(b\,x_\tau - g\,D_{\mathrm{RL}}^\alpha\, x + g\cancel{w})), \quad (9)$$

whose sign is indefinite. However, by using suitable realisations, energy decay can be proven. The proof is split into three steps: hyperbolic realisation of the delay x_τ in § 3.2.1; parabolic realisation of the fractional derivative D_{RL}^α in § 3.2.2; study of the global energy decay in § 3.2.3.

3.2.1 Transport PDE for the time-delay operator Let us realise the delay term through a transport PDE on the 1D spatial domain $z \in (0, \ell)$ with $\ell := c\,\tau$.

$$\partial_t \psi(t, z) = -c\,\partial_z \psi(t, z) \quad (0 < z < \ell) \quad (10)$$

$$\psi(t, z = 0) := x(t) \quad (11)$$

$$x(t - \tau) = \psi(t, z = \ell)\,. \quad (12)$$

The initial data is $\psi(t = \tau, z) := x^0(\tau - {}^{z}/_{c})$. It is natural to examine the energy

$$E_\psi(t) := \frac{1}{2}\int_0^\ell |\psi(t, z)|^2\, \mathrm{d}z\,. \quad (13)$$

A careful computation (all quantities are complex-valued) leads to the following energy balance

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\psi(t) &= -c\int_0^\ell \Re(\partial_z \psi(t, z)\,\overline{\psi}(t, z))\, \mathrm{d}z \quad &(14)\\ &= -\frac{c}{2}\,[\,|\psi(t, z)|^2\,]_0^\ell \quad &(15)\\ &= \frac{c}{2}\,(|x(t)|^2 - |x(t-\tau)|^2)\,. \quad &(16)\end{aligned}$$

3.2.2 Diffusive realisation for the fractional derivative For the Riemann-Liouville fractional derivative, the main idea is to use the diffusive representation, which enables to get an energy balance and help control the cross-product term $\Re(\overline{x}\,D_{\mathrm{RL}}^\alpha x)$ that does not have a definite sign.

A realisation of the Riemann-Liouville fractional derivative is provided in § A.2, see (A.3–A.4). The energy of the additional variables is naturally defined as

$$E_{\widetilde{\varphi}}(t) := \frac{1}{2}\int_0^\infty \xi\,|\widetilde{\varphi}(\xi, t)|^2\, \mu_{\overline{\alpha}}(\xi)\, \mathrm{d}\xi\,. \quad (17)$$

This realisation enjoys the following energy balance (which expresses the dissipativity of D_{RL}^α)

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\widetilde{\varphi}} = \Re(\overline{x}\,D_{\mathrm{RL}}^\alpha x) - \int_0^\infty |x - \xi\,\widetilde{\varphi}(\xi, \cdot)|^2\, \mu_{\overline{\alpha}}(\xi)\, \mathrm{d}\xi\,. \quad (18)$$

3.2.3 Lyapunov stability of the coupled system The energy $\mathcal{E}$ of the coupled system can be built from the three energies encountered above: original E_x, hyperbolic E_ψ, and parabolic E_φ.

$$\mathcal{E}(t) := E_x(t) + k\,E_\psi(t) + g\,E_{\widetilde{\varphi}}(t)\,. \quad (19)$$

The coefficient $k > 0$ is a degree of freedom to be tuned later. The global energy balance reads:

$$\dot{\mathcal{E}} = \dot{E}_x + k\,\dot{E}_\psi + g\,\dot{E}_{\widetilde{\varphi}}. \tag{20}$$

The cross term $g\,\Re(\overline{x}\,D^\alpha_{\mathrm{RL}}x)$ from the original energy balance (9) cancels out with that from the parabolic energy balance (18). The further use of the hyperbolic energy balance (16) leads to

$$\dot{\mathcal{E}} = -X^H\Sigma_k X - g\int_0^\infty |x-\xi\,\widetilde{\varphi}(\xi,\cdot)|^2\mu_{\overline{\alpha}}(\xi)\,\mathrm{d}\xi, \tag{21}$$

where $X := (x, x_\tau)^\intercal$ and

$$\Sigma_k := \begin{pmatrix} \Re(a)+k\dfrac{c}{2} & \dfrac{b}{2} \\ \dfrac{\overline{b}}{2} & -k\dfrac{c}{2}\end{pmatrix}. \tag{22}$$

To conclude the proof, it remains to choose k such that Σ_k is positive definite. Let us first recall the following elementary lemma.

Lemma 6. A 2×2 hermitian matrix $\begin{pmatrix} u & v \\ \overline{v} & w\end{pmatrix}$ is positive definite if and only if $u > 0$ and $|v|^2 < u\,w$.

Proof. It suffices to note that a 2×2 hermitian matrix Σ is positive definite if and only if $\det(\Sigma) > 0$ and $\mathrm{tr}(\Sigma) > 0$. (Product and sum of the two eigenvalues.) □

The hermitian matrix Σ_k is positive definite if and only if $0 < k < -2\Re(a)/c$ and $|b|^2 < kc\,(-2\Re(a) - kc)$. For the optimal value $k^* = -\Re(a)/c$, the least stringent condition on the parameters is obtained: $\Re(a) < -|b| \leq 0$. □

4. A GENERAL RESULT ON STABILITY OF VECTOR-VALUED LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

The general vector-valued model under study is

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) - Gd^\alpha_{\mathrm{C}}x(t) \quad \text{for } t > \tau \tag{23}$$

$$x(t) := x^0(t) \quad \text{for } 0 \leq t \leq \tau. \tag{24}$$

All the matrices involved above are $n\times n$ matrices.

Theorem 7. Let G be a diagonalisable matrix with eigenvalues $(g_1,\cdots,g_n) \geq 0$. Under the algebraic condition

$$\max_{a\in\sigma(A)}\Re(a) < -\sqrt{\max_{b\in\sigma(B^HB)}|b|} \leq 0, \tag{25}$$

the system (23)–(24) with $x^0(0) = 0$ is delay-independent stable. (Here B^H and σ denote the transconjugate of B and point spectrum, respectively.)

Remark 8. If B is diagonalizable (on $\mathbb{C}$), then an elementary computation shows that $\max_{b\in\sigma(B^HB)}|b| = \max_{b\in\sigma(B)}|b|^2$, which is the square of the spectral radius of B. The algebraic condition (25) then reads

$$\max_{a\in\sigma(A)}\Re(a) < -\max_{b\in\sigma(B)}|b| \leq 0.$$

When $G = 0$, several type of sufficient conditions are known, see for instance Kharitonov (1999); Engel and Nagel (2000); Fridman (2014) and references therein. Such a condition typically reads: for any $\Re(s) \geq 0$ and $|z| \geq 1$, $\det(s\,I - A - B\,z^{-1}) \neq 0$. The challenge in verifying such criterion lies in the need to localize zeros. As an alternative, we herein focus on a so-called energy approach that includes the case $G \neq 0$.

Proof. Without loss of generality, we can assume that G is diagonal (otherwise, consider a change of basis on (23)–(24)). The proof is similar in spirit to that of Prop. 4. The delay x_τ is realised as a vector-valued transport equation on ψ (see (10)), while the fractional derivative $D^\alpha_{\mathrm{RL}}x$ is realised through its diffusive representation. The global energy is defined as (contrast with (19))

$$\mathcal{E}(t) := \sum_{i\in[\![1,n]\!]} E_{x_i}(t) + k\,E_{\psi_i}(t) + g_i\,E_{\widetilde{\varphi}_i}(t).$$

Along the trajectories, we get:

$$\dot{\mathcal{E}} = \dot{x}\cdot x + \frac{kc}{2}[\|x\|^2 - \|x(\cdot-\tau)\|^2] + \sum_{i\in[\![1,n]\!]} g_i\dot{E}_{\widetilde{\varphi}_i}. \tag{26}$$

As G is diagonal, there is no coupling between the diffusive variables $\widetilde{\varphi}_i$, and (26) leads to an energy balance akin to (21): the coupling between the delay and fractional operators is straightforward. Since $g_i \geq 0$, it is sufficient for delay-independent stability to prove that

$$-\Sigma_k := \begin{bmatrix} \dfrac{A+A^H}{2} + \dfrac{kc}{2}I & \dfrac{1}{2}B \\ \dfrac{1}{2}B^H & -\dfrac{kc}{2}I \end{bmatrix} < 0,$$

where I denotes the identity matrix on $\mathbb{C}^n$, c is the convection velocity and $k > 0$ is a parameter to be tuned. Let us denote $A^S = {}^{(A+A^H)}/_2$ the symmetric part of A. We have for all x and y in $\mathbb{C}^n$

$$-\Sigma_k\begin{bmatrix}x\\y\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix} = A^Sx\cdot x + \frac{kc}{2}\|x\|^2 + \frac{1}{2}By\cdot x + \frac{1}{2}B^Hx\cdot y - \frac{kc}{2}\|y\|^2.$$

Now, for any $\varepsilon > 0$, from

$$\frac{1}{2}By\cdot x + \frac{1}{2}B^Hx\cdot y = \Re\,(By\cdot x) \leq \frac{\varepsilon}{2}\|By\|^2 + \frac{1}{2\varepsilon}\|x\|^2,$$

and

$$\|By\|^2 = B^HBy\cdot y \leq \max_{b\in\sigma(B^HB)}|b|\|y\|^2,$$

we can choose $k^* = k_\varepsilon = {}^{\varepsilon\max_{b\in\sigma(B^HB)}|b|}/_c > 0$ to get

$$-\Sigma_k\begin{bmatrix}x\\y\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix} \leq \left(\max_{a\in\sigma(A)}\Re(a) + \frac{k_\varepsilon c}{2} + \frac{1}{2\varepsilon}\right)\|x\|^2.$$

Taking the least stringent value of $\varepsilon > 0$, we derive

$$\max_{a\in\sigma(A)}\Re(a) + \sqrt{\max_{b\in\sigma(B^HB)}|b|} < 0$$

as a delay-independent stability sufficient condition. □

Remark 9. If the three matrices A, B and G are assumed simultaneously diagonalizable, the proof of Thm. 7 can be performed directly in the Laplace domain, as for the toy model above. However, the Laplace transform does not appear fruitful to deal with the general case.

Remark 10. The proof breaks down if the delay and fractional operator are *composed*, i.e. $d^\alpha x(t-\tau)$ in (23). Nonetheless, this case can be tackled numerically, see § 5.

5. AN EIGENVALUE APPROACH TO STABILITY OF FRACTIONAL DELAY SYSTEMS

The method of proof, based on parabolic-hyperbolic realisations, naturally suggests a numerical method to discretise (notice the composition for $\tau_\alpha \neq 0$)

$$\dot{x}(t) = A\,x(t) + B\,x(t-\tau) - G\,d^\alpha_C x(t-\tau_\alpha) \tag{27}$$

into the Cauchy problem on $\mathbb{C}^n$

$$\dot{X}_h(t) = \mathcal{A}_h\,X_h(t), \tag{28}$$

where $X_h := (x, \psi_h, \varphi_h)$ is the extended state. $\mathcal{A}_h$ is a finite-dimensional approximation of the semigroup generator $\mathcal{A}$ (whose existence is herein assumed). To study the stability of (27) using $\sigma(\mathcal{A}_h)$, we assume spectral convergence, without lack of approximation or spectral pollution. A numerical method enables to both check the stability conditions provided above and investigate more intricate systems where it is the only way to conclude. An efficient numerical method will be presented at the conference; some early results are presented below.

Spectral structure. Fig. 2 plots $\sigma(\mathcal{A})$, with A and B chosen to verify the condition (25) of Thm. 7. The spectrum has two components: an *essential* one on $(-\infty, 0)$, associated with the diffusive representation of the fractional derivative; a *discrete* one, typical of time-delay operators. Setting $g > 0$ ($g < 0$) has a (de)stabilizing effect.

Delay-dependent stability. The scalar case is convenient to perform a parametric study with a complex-valued g. Thm. 2 states that for (a,b) that verify the condition (7), delay-independent stability is achieved for g within the sector J_α. Nonetheless, the τ-dependent stability region is larger than this sector. This is illustrated in Fig. 3 that shows, for a given modulus $|g|$ and delay τ, the angle θ_g above which (5) is unstable, denoted $\theta_g^{\max}$.

Composition. Although the presented proofs break down for $\tau_\alpha \neq 0$, this case can be studied numerically. Fig. 4 illustrates the impact of setting $\tau_\alpha = \tau$ on the spectrum. Here, qualitatively, τ_α has a stabilising effect for $g = {}^{|a|}/_4$,

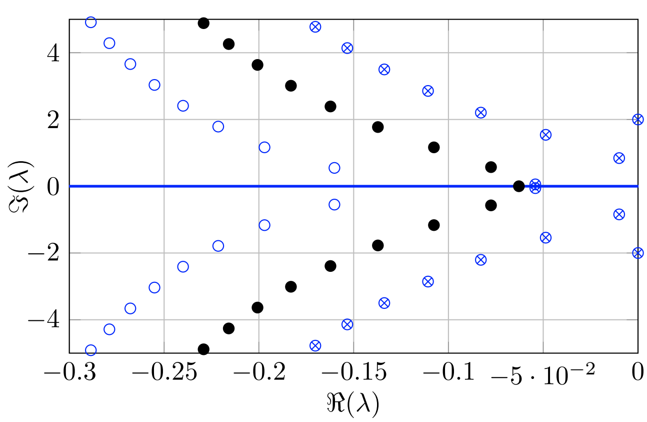

Fig. 2. Influence of g on the spectrum $\sigma(\mathcal{A})$ for (23) with $(\alpha, \tau) = (\frac{1}{2}, 10)$, $A = \frac{1}{2}\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$, $B = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $G = g\,I$. (●): $g = 0$. (○): $g = +2$. (⊗): $g = -2$. (——): essential spectrum ($g \neq 0$ only).

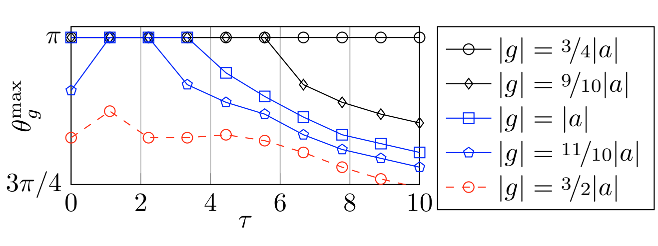

Fig. 3. τ-dependent stability region for (27) with parameters $\alpha = {}^1/_2$, $a = -1$, $b = {}^{|a|}/_2$, and $g = |g|e^{i\theta_g}$.

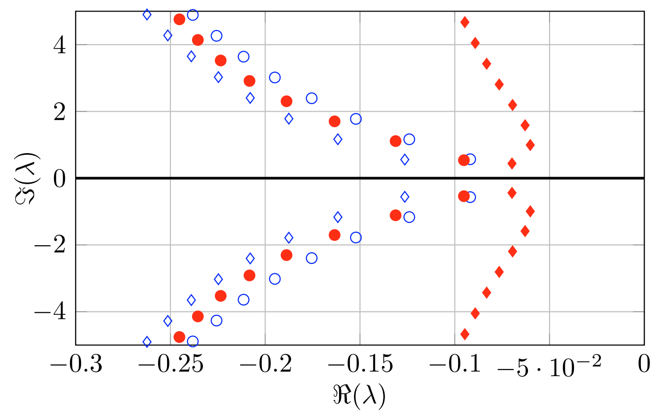

Fig. 4. Influence of τ_α on the spectrum $\sigma(\mathcal{A})$, with (a, b) same as for Fig. 3 and $\tau = 10$. (◊): $(g, \tau_\alpha) = (|a|, 0)$. (♦): $(g, \tau_\alpha) = (|a|, 10)$. (○): $(g, \tau_\alpha) = ({}^{|a|}/_4, 0)$. (●): $(g, \tau_\alpha) = ({}^{|a|}/_4, 10)$. (——): essential spectrum (common to the four cases).

while the reverse is true for a larger value $g = |a|$ (in contrast with the phenomenon highlighted in Fig. 2).

6. EXTENSIONS AND OPEN QUESTIONS

6.1 Multiple delay case

A straightforward extension of Thm. 7 is the case of multiple, *non-commensurate* delays. Each delay $x(t - \tau_j)$ is realised through its own transport equation: in (10), a diagonal velocity matrix C then comes into play.

6.2 Semigroup formulation

The proofs of stability given in Prop. 4 and Thm. 7 would be complete if one could invoke LaSalle's invariance principle (Cazenave and Haraux, 1998, Thm. 9.2.3). However, its rigorous application to infinite-dimensional systems requires to check a priori the precompactness of the trajectories in the energy space. This is not a mere formality: the diffusive representation does induce a lack of precompactness, as pointed out in Matignon and Prieur (2005) for an ODE with fractional derivative, and in Matignon and Prieur (2014) for a PDE with fractional derivative. In both these references, the authors turn to the Arendt-Batty stability theorem to prove the asymptotic stability of the coupled system, which requires a refined analysis of the spectrum of the generator of the underlying semigroup. Similar techniques applied to the operator $\mathcal{A}$ that generates the coupled system (23) can be expected to yield well-posedness and, under the given algebraic conditions, asymptotic stability.

6.3 Time-domain method with the Caputo derivative

If $x^0(0) \neq 0$, then d_C^α is different from D_{RL}^α, see (B.1). As a result, $w \neq 0$ in (9) and the given proofs for Prop. 4 and Thm. 7 break down (i.e. energy decay is not achieved using the diffusive representation of D_{RL}^α (A.3)–(A.4)). To circumvent this, one may consider directly using the diffusive representation of d_C^α given by (B.3)–(B.4). However, since $\varphi(\cdot, 0) \neq 0$, the energy (17) is then *infinite*:

the energy method still breaks down. A suitable energy space is proposed in § B.3; nonetheless, this norm does not yield an energy balance analogous to (18).
These difficulties stem from the fact that the Caputo derivative is *not* a dissipative operator (see (Lozano et al., 2007, Def. 4.7)) if $x^0(0) \neq 0$, hence the breakdown of the energy method. To the best of the authors' knowledge, a time-domain method of proof that do not need the assumption $x^0(0) = 0$ is unavailable.

6.4 Eigenvalue approach to stability

As pointed out in § 5, the numerical method can be used to investigate the stability of more intricate cases. Moreover, the spectral properties of the numerical method have only been assumed so far. This could be further investigated.

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Appendix A. FRACTIONAL DERIVATIVES AND DIFFUSIVE REPRESENTATIONS

A.1 Fractional integral

Let $u \in L^2(0,T)$, for any $\beta \in (0, 1)$, define the causal convolution kernel $h_\beta(t) := \Gamma(\beta)^{-1} t^{\beta-1}$ for $t > 0$, then the fractional integral of order $\beta \in (0, 1)$ of u is defined by $I^\beta u := h_\beta \star u$.

Since $h_\beta(t) = \int_0^\infty \mu_\beta(\xi)\, e^{-\xi\, t}\, \mathrm{d}\xi$, with specific weight $\mu_\beta(\xi) = \pi^{-1} \sin(\beta\,\pi)\, \xi^{-\beta}$, the fractional integral can be reformulated by the following input-output representation

$$y(t) = \int_0^\infty \mu_\beta(\xi)\, [e_\xi \star u](t)\, \mathrm{d}\xi\,,$$

with $e_\xi(t) := e^{-\xi\, t}$, and $[e_\xi \star u](t) = \int_0^t e^{-\xi\,(t-\tau)}\, u(\tau)\, \mathrm{d}\tau$. The following infinite-dimensional dynamical system can be seen as a state-space realisation of the fractional integral of order β

$$\partial_t \varphi(\xi, t) = -\xi\, \varphi(\xi, t) + u(t),\ \ \varphi(\xi, 0) = 0\,, \qquad \text{(A.1)}$$

$$y(t) = \int_0^\infty \mu_\beta(\xi)\, \varphi(\xi, t)\, \mathrm{d}\xi\,. \qquad \text{(A.2)}$$

The functional state space for the diffusive variables $\varphi(\cdot, t)$ is $\mathcal{H}_\beta = L^2(\mathbb{R}^+, \mu_\beta\, \mathrm{d}\xi)$, see e.g. Matignon and Prieur

(2014). It defines a well-posed linear system, see Matignon and Zwart (2004).

A.2 Riemann-Liouville fractional derivative

For short, the Riemann-Liouville fractional *derivative* of order $\alpha \in (0,1)$ of $u \in H^1(0,T)$ is defined by $\widetilde{y} = D^\alpha u = D[I^{1-\alpha}u]$, and a careful computation shows that the following input-output representation holds

$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u - \xi\, e_\xi \star u](t)\,\mathrm{d}\xi\,.$$

The following infinite-dimensional dynamical system can be seen as a state-space realisation of the fractional derivative of order α

$$\partial_t \widetilde{\varphi}(\xi,t) = -\xi\,\widetilde{\varphi}(\xi,t) + u(t),\ \widetilde{\varphi}(\xi,0) = 0\,, \tag{A.3}$$

$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u(t) - \xi\,\widetilde{\varphi}(\xi,t)]\,\mathrm{d}\xi\,. \tag{A.4}$$

The functional state space for the diffusive variables $\widetilde{\varphi}(\cdot,t)$ is $\widetilde{\mathcal{H}}_\alpha = L^2(\mathbb{R}^+, \xi\,\mu_{1-\alpha}\,\mathrm{d}\xi)$, see e.g. Matignon and Prieur (2014). Compared with, for instance, the Grünwald-Letnikov formula, the computational interest of (A.3)–(A.4) is that it is a *time-local* representation of the hereditary operator D^α.

Appendix B. CAPUTO FRACTIONAL DERIVATIVE AND INITIAL CONDITIONS

B.1 Caputo and Riemann-Liouville fractional derivatives

For a function $u \in H^1(0,T)$, $D^\alpha_{\mathrm{RL}} u := \frac{d}{dt} I^{1-\alpha}[u]$, whereas $d^\alpha_{\mathrm{C}} u := I^{1-\alpha}[\dot{u}]$. A careful computation shows the following relation between the two fractional derivatives

$$d^\alpha_{\mathrm{C}} u(t) = D^\alpha_{\mathrm{RL}} u(t) - u(0^+)\,\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \tag{B.1}$$

showing more regularity for d^α_{C} than D^α_{RL}, as pointed out in Matignon (2009). Since the infinite-dimensional dynamical system (A.3)–(A.4) provides a realisation for D^α_{RL}, it thus proves necessary to adapt to d^α_{C}. Hopefully, since only the initial condition on u does play a role, a slight modification of the initial state $\widetilde{\varphi}(.,0)$ in (A.3) will do. Indeed, following e.g. (Lombard and Matignon, 2016, Prop. 2.1), we find that $\widetilde{\varphi}^0(\xi) := {}^{u(0^+)}/_{\xi}$ is the appropriate initial datum. However, since $\widetilde{\varphi}^0 \notin \widetilde{\mathcal{H}}_\alpha$, there is a theoretical need to extend the framework, see § B.3.

B.2 Making use of the Callier-Desoer class

The proof of Thm. 2 relies heavily on Thm. A.7.49 of Curtain and Zwart (1995). Let us denote

$$\hat{h}(s) := (s - a - b\,\exp(-\tau\,s) + g\,s^\alpha)^{-1}\,,$$

we want to prove that $\hat{h} \in \widehat{\mathcal{A}}(0)$, the Callier-Desoer class of transfer functions, when $\Re(a) < -|b| \le 0$ and $g \in J_\alpha := [-\overline{\alpha}\,\frac{\pi}{2}, \overline{\alpha}\,\frac{\pi}{2}]$.

First, $(s-a)^{-1} \in \widehat{\mathcal{A}}(0)$, since $\Re(a) < 0$. It remains to show that $[1 - (b\,\exp(-\tau\,s) - g\,s^\alpha)/(s-a)]^{-1} \in \widehat{\mathcal{A}}(0)$. We first notice that $\hat{f}(s) = 1 - (b\,\exp(-\tau\,s) - g\,s^\alpha)/(s-a) \in \widehat{\mathcal{A}}(0)$. Then, it is enough to prove that $\inf_{\Re(s)\ge 0} |\hat{f}(s)| > 0$ to ensure that $1/\hat{f} \in \widehat{\mathcal{A}}(0)$ also. Finally, the product of transfer functions is stable in $\widehat{\mathcal{A}}(0)$.

Indeed, since $\inf_{\Re(s)\ge 0} |s-a| = -\Re(a) > 0$, we rather compute $(s-a)\,\hat{f}(s) = s - a - b\,\exp(-\tau\,s) + g\,s^\alpha$, and examine its real part with $s = x + i\,y = r\,\exp(i\theta)$, when $\{x \ge 0 \text{ and } y \in \mathbb{R}\}$, or $\{r \ge 0 \text{ and } |\theta| \le \frac{\pi}{2}\}$. Recall, from the proof of Thm. 2, that

$$\Re((s-a)\,\hat{f}(s)) \ge -\Re(a) - |b| > 0\,,$$

where have used $\theta_g \in J_\alpha := [-\overline{\alpha}\,\frac{\pi}{2}, \overline{\alpha}\,\frac{\pi}{2}]$, hence $\forall\,|\theta| \le \frac{\pi}{2}$, $\cos(\theta_g + \alpha\,\theta) \ge 0$.

As a consequence, $h \in \mathcal{A}(0)$, i.e. can be decomposed into

$$h(t) = h_a(t) + \sum_{n\in\mathbb{N}} h_n\,\delta(t - t_n)\,, \tag{B.2}$$

with $h_a \in L^1(\mathbb{R}^+)$ and $(h_n)_{n\in\mathbb{N}} \in l^1(\mathbb{N})$; and $0 = t_0 < t_1 < t_2 < \cdots$. Then, the Lebesgue dominated convergence theorem can be applied, and for any $w \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, if $\lim_{t\to\infty} w(t) = 0$, then $(h \star w)(t) \to 0$, as $t \to \infty$. We particularise the computation with $w(t) \propto t^{-\alpha}$. □

B.3 Diffusive representation of fractional derivatives of Caputo type with non-null initial condition

The following infinite-dimensional dynamical system can be seen as a state-space realisation of the Caputo fractional derivative of order α

$$\partial_t \check{\varphi}(\xi,t) = -\xi\,\check{\varphi}(\xi,t) + u(t),\ \check{\varphi}(\xi,0) = {}^{u(0^+)}/_{\xi}\,, \tag{B.3}$$

$$\check{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u(t) - \xi\,\check{\varphi}(\xi,t)]\,\mathrm{d}\xi\,. \tag{B.4}$$

The main difficulty is to give a functional framework such that these equations make sense. Indeed, as already said, $\check{\varphi}(\xi,0) = {}^{u(0^+)}/_{\xi} \notin \widetilde{\mathcal{H}}_\alpha$. Let us now define

$$\check{\mathcal{H}}_\alpha = L^2\left(\mathbb{R}^+, \frac{\xi^2}{1+\xi}\,\mu_{1-\alpha}\,\mathrm{d}\xi\right),$$

and

$$\check{\mathcal{H}}_{\alpha,-1} = L^2\left(\mathbb{R}^+, \frac{\xi^2}{(1+\xi)(1+\xi^2)}\,\mu_{1-\alpha}\,\mathrm{d}\xi\right).$$

Then, one can show that for all inputs $u \in H^1(0,T)$, $\check{\varphi}(\xi,0) = {}^{u(0^+)}/_{\xi} \in \check{\mathcal{H}}_\alpha$ and $\mathbb{1}(\xi)u(t) \in H^1_{\mathrm{loc}}((0,\infty), \check{\mathcal{H}}_{\alpha,-1})$. Finally from Engel and Nagel (2000) and (Tucsnak and Weiss, 2009, Thm. 4.1.6), there exists a unique solution to (B.3) satisfying $\check{\varphi} \in C([0,T]; \check{\mathcal{H}}_\alpha) \cap C^1([0,T); \check{\mathcal{H}}_{\alpha,-1})$. From straightforward computation and the following Duhamel's formula

$$\check{\varphi}(\xi,t) = e^{-\xi t}\,\frac{u(0^+)}{\xi} + \int_0^t e^{-\xi(t-\tau)}u(\tau)\,\mathrm{d}\tau\,,$$

one can easily check, by integration by parts, that

$$[e_\xi \star \dot{u}](t) = u(t) - \xi\,\check{\varphi}(\xi,t)\,,$$

which implies for (B.4) that

$$\begin{aligned} \check{y}(t) &= \int_0^\infty \left(\int_0^t e^{-\xi(t-\tau)}\dot{u}(\tau)\,\mathrm{d}\tau\right) \mu_{1-\alpha}(\xi)\,\mathrm{d}\xi\,, \\ &= \int_0^\infty \mu_{1-\alpha}(\xi)\,[e_\xi \star \dot{u}](t)\,\mathrm{d}\xi\,, \\ &= \left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)} \star \dot{u}\right](t)\,. \end{aligned}$$

The last equality comes from Fubini theorem, and the result means that $\check{y} = I^{1-\alpha}[\dot{u}] := d^\alpha_C u$ indeed. Then, since $u \in H^1(0,T)$, we can easily see that for all $\alpha \in (0,1)$, $\check{y} \in L^2(0,T)$; thus, the output makes sense.