Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 50-1 (2017) 4546–4551

Closed-loop perturbations of well-posed linear systems

Ghislain Haine *

* *Institut Supérieur de l'Aéronautique et de l'Espace ISAE-SUPAERO, Université de Toulouse, 31055 TOULOUSE Cedex 4*

Abstract: We are concerned with the perturbation of a rather general class of linear time-invariant systems, namely *well-posed linear system* (WPLS), under additive linear perturbations seen as feedback laws. Let Σ be a WPLS with (A, B, C) as generating triple. For all control operator E, and all observation operator F such that (A, E, F) is the generating triple of a WPLS, we prove that, if (A, B, F) and (A, E, C) are also the generating triples of some WPLS, for all admissible feedback operator K for (A, E, F), we can construct a WPLS Σ^K whose generating triple is (A^K, B^K, C^K), where A^K is the infinitisemal generator of the closed-loop of (A, E, F) by the feedback operator K. Furthermore, we give necessary and sufficient condition such that exact controllability persists from Σ to Σ^K. In particular, we show that this is the case for all sufficiently small bounded operator K.

Keywords: Linear systems, Closed-loop systems, Perturbation, Controllability, Robustness.

1. INTRODUCTION

In this work, we are interested in the robustness of exact controllability and exact observability of well-posed linear systems under some type of perturbations. Due to lack of space, we will only talk about exact controllability. However, since we are dealing in the sequel with Hilbert spaces, exact controllability is dual to exact observability, see for instance (Tucsnak and Weiss, 2009, Chapter 11), and similar results can be easily obtained for exact observability. These concepts are supposed to be known, but exact controllability will be properly defined in Section 2.

Well-posed linear systems are now a well-known class of linear time-invariant systems, which has been introduced in their modern form in the late 80's (see the work of Salamon (1987, 1989); Weiss (1989a,b, 1994b); Curtain and Weiss (1989)). This class allows to write a wide range of partial differential equations in abstract form. Other more general classes of linear time-invariant systems have been investigated, such as system nodes, see Staffans (2005); Tucsnak and Weiss (2014), or even more general as resolvent linear systems, see Opmeer (2005). For more details on well-posed linear systems, we refer to the survey papers by Weiss et al. (2001); Tucsnak and Weiss (2014) and the book by Staffans (2005).

A well-known result in finite dimension (Lee and Markus, 1967, Th. 11 p. 100) says that, to quote: *The set of all controllable processes is open and dense in the metric of all autonomous linear processes in* $\mathbb{R}^n$. This means in particular that for all small enough perturbations of a controllable system, controllability persists. In the infinite dimensional setting, there is a very wide literature about this subject. Among these works, we can cite Leiva (2003); Boulite et al. (2005); Hadd (2005); Mei and Peng (2010); Cîndea and Tucsnak (2010); Mei and Peng (2014). Except for some particular partial differential equations where the results can be stronger (this is the case for instance in (Cîndea and Tucsnak, 2010, Theorem 1)), they all conclude that for all *small enough* perturbations, exact controllability persists. But the "small enough" has to be understood "small enough in the class where we allow the control operator and the perturbation to lie in". The aim of this work is to extend the results in Hadd (2005); Mei and Peng (2010), by allowing a more general class of perturbations.

1.1 Closed-loop perturbations

To make easier to understand which type of perturbations we have in mind, let us focus on finite dimensional linear time-invariant systems for this subsection. A linear time-invariant system can be represented by four matrices A, B, C, and D, with appropriate dimensions, such that the *control* u, the *state* z, and the *observation* y satisfy

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & \forall t \geq 0, \\ y(t) = Cz(t) + Du(t), & \forall t \geq 0, \end{cases} \tag{1}$$

together with the initial condition $z(0) = z_0$.

To consider additive linear perturbations in system (1), we have at least two ways. On the one hand, one can consider A alone, adds a linear pertubation P (that is a matrix with appropriate dimensions), and look at the system whose four matrices are $A + P$, B, C, and D. However, it can then be difficult to link the initial system to the perturbed one, and hence their respective properties. On the other hand, one could consider perturbations as feedback laws: this is the idea we follow in this work, borrowed from Hadd (2005); Mei and Peng (2010). The main drawback of this approach is that in general we then perturb the whole system, that is A, B, C, and D, and not only A. The advantage is that we easily link the initial system to the

2405-8963 © 2017, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2017.08.729

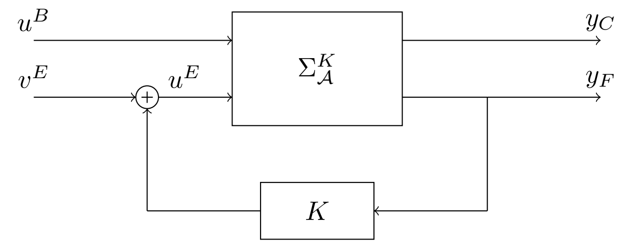

Fig. 1. The augmented system perturbed by a closed loop.

perturbed one but, *a priori, we do not control nor observe the same way after perturbation.*

More precisely let A, B, E, C, F, $D_C^B = D$, D_F^B, D_C^E, and D_F^E, be nine matrices (with appropriate dimensions), we then consider the *augmented system*

$$\begin{cases} \dot{z}(t) = Az(t) + Bu^B(t) + Eu^E(t), \\ y_C(t) = Cz(t) + D_C^B u^B(t) + D_C^E u^E(t), \\ y_F(t) = Fz(t) + D_F^B u^B(t) + D_F^E u^E(t), \end{cases}$$

together with the initial condition $z(0) = z_0$. It is clear that if we take $u^E \equiv 0$ and "forget" y_F, we come back to system (1).

Now our perturbation is given by considering the feedback law

$$u^E(t) = Ky_F(t) + v^E(t), \quad \forall t \geq 0,$$

where K is an appropriate feedback matrix and v^E is the new input function for E.

We can see on Fig. 1 a representation of the closed augmented system.

Alternatively, this perturbation can be described as follows:

$$\mathcal{C} = \begin{bmatrix} C \\ F \end{bmatrix}, \quad \mathcal{B} = [B \ \ E],$$

$$\mathcal{D} = \begin{bmatrix} D_C^B & D_C^E \\ D_F^B & D_F^E \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix},$$

and finally

$$y = \begin{bmatrix} y_C \\ y_F \end{bmatrix}, \quad u = \begin{bmatrix} u^B & u^E \end{bmatrix} = \mathcal{K}y + v,$$

we then have a classic closed-loop by $\mathcal{K}$ of system whose matrices are A, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. Hence, an easy exercise if $\mathcal{K}$ is admissible, *i.e.* if $I - \mathcal{D}\mathcal{K}$ is invertible (equivalently $I - \mathcal{K}\mathcal{D}$) (we denote the identity on any space by the same symbol I), gives us that the closed-loop system's matrices are given by

$$A^K = A + \mathcal{B}\mathcal{K}(I - \mathcal{D}\mathcal{K})^{-1}\mathcal{C}, \quad \mathcal{B}^K = \mathcal{B}(I - \mathcal{K}\mathcal{D})^{-1},$$
$$\mathcal{C}^K = (I - \mathcal{D}\mathcal{K})^{-1}\mathcal{C}, \quad \mathcal{D}^K = (I - \mathcal{D}\mathcal{K})^{-1}\mathcal{D}.$$

This is also an easy exercise to see that in fact, $\mathcal{K}$ is admissible (for $\mathcal{D}$) if and only if K is admissible for D_F^E, *i.e.* $I - KD_F^E$ and $I - D_F^E K$ are invertible.

Finally, some basic computations allow us, after taking $v^E \equiv 0$ and "forgetting" y_F, to come back to a perturbed system closely related to the unperturbed one (1), whose matrices are given by

$$A^K = A + EK(I - D_F^E K)^{-1}F,$$
$$B^K = B + E(I - KD_F^E)^{-1}KD_F^B,$$
$$C^K = C + D_C^E K(I - D_F^E K)^{-1}F,$$
$$D^K = D_C^B + D_C^E K(I - D_F^E K)^{-1}D_F^B.$$

In particular, we can appreciate that we finally construct a system where the additive perturbation of A is the one corresponding to the closed-loop of A, E, F, D_F^E by K, while our new control and observation matrices are just additive linear perturbations of B and C.

Remark 1. In this finite dimensional situation, D_F^B, D_C^E, and D_F^E can be choosen equal to zero without compromising the *augmentation* step in the process since the resulting systems will always be well-posed. Then we have $A^K = A + EKF$, $B^K = B$, $C^K = C$, and $D^K = D$. However, this is not the case in general. In fact, we need Hypothesis 13 (this is a necessary and sufficient condition) to do the augmentation step in general. This is why we give the general form even for the finite dimensional setting.

1.2 Well-posed linear systems

Thanks to Weiss (1994a), the idea of closed-loop perturbation can be extended directly to a wide class of infinite-dimensional systems: *well-posed linear systems.* Now A, B, C, E, and F are possibly unbounded operators (while operators Ds may be non-unique, depending on choices to continuously extend operators C and F on larger spaces, but they are bounded), and without any additional assumptions, we are no longer able to write down *easily* the operators that generate the perturbed system.

Roughly speaking, well-posed linear systems are the generalization of (1), in the integrated form

$$\begin{cases} z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A}Bu(s)ds, \\ y(t) = Ce^{tA}z_0 + C\int_0^t e^{(t-s)A}Bu(s)ds + Du(t), \end{cases}$$

to infinite dimensional spaces U, X, and Y.

1.3 Outline of the paper

This paper is organized as follows. In Section 2, we recall the basics of well-posed linear systems. In Section 3, we augment the initial system with the operators that describe the perturbation to another well-posed linear system, define the feedback law which gives the closed-loop system, as in the finite dimensional situation, to finally get our perturbed well-posed linear system. Hypothesis 13 appears naturally in this part. In Section 4, we give our main results about the robustness of exact controllability under small perturbations, namely Corollary 16. Finally, in Section 5, we specify our result to the regular cases considered in Hadd (2005); Mei and Peng (2010).

2. BACKGROUND ON WELL-POSED LINEAR SYSTEMS

All the material recall in this section can be found in Weiss et al. (2001) and the references therein.

2.1 Definition

We first define the *τ-concatenation.* For any $\tau \geq 0$ and any Z, Hilbert space, we define for all u, v in $L^2([0,\infty), Z)$ the following binary operator

$$\left(u \underset{\tau}{\diamond} v\right)(t) = \begin{cases} u(t), & \forall t \in [0,\tau], \\ v(t-\tau), & \forall t \geq \tau. \end{cases}$$

Denoting $\mathbf{P}_\tau$ the projection of $L^2([0,\infty), Z)$ on $L^2([0,\tau), Z)$ (by truncation) and $\mathbf{S}_\tau$ the right shift (by τ) operator on $L^2([0,\infty), Z)$, we can rewrite

$$u \underset{\tau}{\diamond} v = \mathbf{P}_\tau u + \mathbf{S}_\tau v.$$

Definition 2. (Well-Posed Linear System). Let U, X, and Y be three Hilbert spaces, called respectively the *input space*, the *state space*, and the *output space*. We denote by $\mathcal{U} = L^2((0,\infty); U)$ and $\mathcal{Y} = L^2((0,\infty); Y)$. A well-posed linear system $\Sigma = (\Sigma_t)_{t\geq 0}$ on (U, X, Y) is a quadruplet $\Sigma = \begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix}$ such that

(1) $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ is a C_0-semigroup on X,
(2) $\Phi = (\Phi_t)_{t\geq 0}$ is a family of bounded linear operators from $\mathcal{U}$ to X such that
$$\Phi_{\tau+t}\left(u \diamond_\tau v\right) = \mathbb{T}_t \Phi_\tau u + \Phi_t v, \quad \forall u, v \in \mathcal{U},\ \tau, t \geq 0,$$
(3) $\Psi = (\Psi_t)_{t\geq 0}$ is a family of bounded linear operators from X to $\mathcal{Y}$ such that
$$\Psi_{\tau+t} z = \Psi_\tau z \diamond_\tau \Psi_t \mathbb{T}_\tau z, \quad \forall z \in X,\ \tau, t \geq 0,$$
and $\Psi_0 \equiv 0$,
(4) $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$ such that for all u and $v \in \mathcal{U}$ and all $\tau, t \geq 0$, we have
$$\mathbb{F}_{\tau+t}\left(u \diamond_\tau v\right) = \left(\mathbb{F}_\tau u\right) \diamond_\tau \left(\Psi_t \Phi_\tau u + \mathbb{F}_t v\right),$$
and $\mathbb{F}_0 \equiv 0$.

The operators Φ_τ are called *input maps*, Ψ_τ are called *output maps* and $\mathbb{F}_\tau$ are called *input–output maps*.

Definition 3. (Exact controllability). Let $\tau > 0$, we say that a well-posed linear system $\Sigma = \begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix}$ is *exactly controllable in time* τ if and only if $\operatorname{Ran} \Phi_\tau = X$, where Ran means the range of the opeartor.

2.2 Realization

Let A be the infinitesimal generator of a semigroup $\mathbb{T}$ on X, and $\beta \in \rho(A)$ a fixed number in the resolvent set of A. We denote X_1 the Hilbert space obtained when $\mathcal{D}(A)$ is endowed with the norm $\|z_0\|_1 := \|(\beta I - A) z_0\|$ for $z_0 \in \mathcal{D}(A)$, where $\|\cdot\|$ without subscript denote the norm in X. Let us also define X_{-1} as the completion of X by the norm $\|z_0\|_{-1} := \|(\beta I - A)^{-1} z_0\|$, for $z_0 \in X$. Then X_{-1} is a Hilbert space and we have

$$X_1 \subset X \subset X_{-1},$$

with dense and continuous embedding.

Theorem 4. (Generating triple). Let Σ be a well-posed linear system on (U, X, Y). Then

- There exist a unique *control operator* $B \in \mathcal{L}(U, X_{-1})$ such that
$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s) ds, \quad \forall t \geq 0, u \in \mathcal{U}. \qquad (2)$$
- There exist a unique *observation operator* $C \in \mathcal{L}(X_1, Y)$ such that for all $z_0 \in X_1$
$$\Psi_t z_0 = \begin{cases} C \mathbb{T}_s z_0, & 0 \leq s \leq t, \\ 0, & s \geq 0. \end{cases} \qquad (3)$$

The triple (A, B, C) is called the *generating triple* of Σ.

Definition 5. (Well-posed triple). Let A be an infinitesimal generator, $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(X_1, Y)$. We say that the triple (A, B, C) is well-posed on (U, X, Y) if it is the generating triple of a well-posed linear system Σ on (U, X, Y).

Definition 6. (Transfer function). There exists a unique analytic $\mathcal{L}(U, Y)$-valued *well-posed*, *i.e.* its domain contains a right half-plane, function $\mathbf{G}$ called the *transfer function* of Σ, which determines $\mathbb{F}$. If $z_0 = 0$ and $u \in \mathcal{U}$, then $y = \mathbb{F} u$ and y has a Laplace transform $\widehat{y}$ such that for all $s \in \mathbb{C}$ with sufficiently large real part:

$$\widehat{y}(s) = \mathbf{G}(s) \widehat{u}(s).$$

Furthermore we have for all $s, \beta \in \{s \in \mathbb{C} | \operatorname{Re}(s) > \omega_0(\mathbb{T})\}$, where $\omega_0(\mathbb{T})$ is the growth bound of the semigroup $\mathbb{T}$,

$$\mathbf{G}(s) - \mathbf{G}(\beta) = C\left[(sI - A)^{-1} - (\beta I - A)^{-1}\right] B. \qquad (4)$$

Definition 7. (Admissibility).

- We say that a control operator $B \in \mathcal{L}(U, X_{-1})$ is *admissible* for the semigroup $\mathbb{T}$ if for some (hence all) $t > 0$, the operator $\Phi_t : \mathcal{U} \to X_{-1}$ defined by (2) has its range in X.
- We say that an observation operator $C \in \mathcal{L}(X_1, Y)$ is *admissible* for the semigroup $\mathbb{T}$ if for some (hence all) $t > 0$, the operator $\Psi_t : X_1 \to \mathcal{Y}$ defined by (3) has a continuous extension to X.

Theorem 8. A triple of operators (A, B, C) is well-posed on (U, X, Y) if and only if

(1) A is the generator of a strongly continuous semigroup on X.
(2) $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$.
(3) $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$.
(4) there is an $\alpha \in \mathbb{R}$ such that some (hence any) solution $\mathbf{H} : \rho(A) \to \mathcal{L}(U, Y)$ of the equation (4) is bounded on $\{s \in \mathbb{C} | \operatorname{Re}(s) > \alpha\}$.

2.3 Feedback

Definition 9. (Admissible feedback). Let Σ be a well-posed linear system on (U, X, Y). An operator $K \in \mathcal{L}(Y, U)$ is an *admissible (static) output feedback operator* for Σ (or for $\mathbf{G}$) if $I - \mathbf{G}K$ has a well-posed inverse (equivalently, if $I - K\mathbf{G}$ has a well-posed inverse), *i.e.* bounded on some right half-plane.

From (Weiss, 1994a, Definition 3.11, Proposition 3.12), we get the following useful result.

Proposition 10. For each well-posed linear system Σ, there exists a $d \in (0, \infty]$ such that all $K \in \mathcal{L}(Y, U)$ with $\|K\|_{\mathcal{L}(Y,U)} < d$ is an admissible output feedback operator. The greatest d is called the *well-posedness radius* of $\mathbf{G}$, the transfer function of Σ.

Proposition 11. If K is admissible for Σ, then the feedback law $u = Ky + v$ determines a new well-posed linear system Σ^K, called the *closed-loop system*, unique solution to

$$\Sigma_\tau^K - \Sigma_\tau = \Sigma_\tau \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma_\tau^K, \quad \forall \tau > 0. \qquad (5)$$

Furthermore, we have the commutation property

$$\Sigma_\tau \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma_\tau^K = \Sigma_\tau^K \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma_\tau, \quad \forall \tau > 0. \tag{6}$$

From (5)–(6), we directly get that

$$(I - \mathbb{F}_\tau K)(I + \mathbb{F}_\tau^K K) = I = (I + \mathbb{F}_\tau^K K)(I - \mathbb{F}_\tau K), \tag{7}$$

and

$$(I - K\mathbb{F}_\tau)(I + K\mathbb{F}_\tau^K) = I = (I + K\mathbb{F}_\tau^K)(I - K\mathbb{F}_\tau). \tag{8}$$

From (5), we also get

$$\Phi_\tau = \Phi_\tau^K (I + K\mathbb{F}_\tau^K). \tag{9}$$

3. CLOSED-LOOP PERTURBATIONS

Suppose that Σ_C^B and Σ_F^E are well-posed linear systems on (U^B, X, Y_C) and (U^E, X, Y_F) respectively, with respective generating triple (A, B, C) and (A, E, F). We would like to obtain a well-posed linear system Σ^K whose infinitesimal generator is the one obtained by a closed-loop of Σ_F^E by an output feedback operator $K \in \mathcal{L}(Y_F, U^E)$, while control and observation operators are closely related to B and C. The idea is to follow Fig 1. Obviously, this means that $(A, \mathcal{B}, \mathcal{C})$, with $\mathcal{B} = [B \ E]$ and $\mathcal{C} = \begin{bmatrix} C \\ D \end{bmatrix}$, has to be well-posed on $(U^B \times U^E, X, Y_C \times Y_F)$ in the sense of Definition 5.

Lemma 12. The triple of operators $(A, \mathcal{B}, \mathcal{C})$ is well-posed on $(U^B \times U^E, X, Y_C \times Y_F)$ if and only if the four following assertions are fulfilled:

- (A, B, C) is well-posed on (U^B, X, Y_C),
- (A, E, F) is well-posed on (U^E, X, Y_F),
- (A, E, C) is well-posed on (U^E, X, Y_C),
- (A, B, F) is well-posed on (U^B, X, Y_F).

Proof. By straightforward computation, we see that

$$\mathbb{F} := \begin{bmatrix} \mathbb{F}^{BC} & R \\ S & \mathbb{F}^{EF} \end{bmatrix},$$

where R and S are families of bounded linear operators from $\mathcal{U}^E$ to $\mathcal{Y}_C$, and from $\mathcal{U}^B$ to $\mathcal{Y}_F$ respectively, is the input–output map of a well-posed linear system with $\mathbb{T}$ as semigroup, $\Phi = \begin{bmatrix} \Phi^B & \Phi^E \end{bmatrix}$ as input map and $\Psi = \begin{bmatrix} \Psi^C \\ \Psi^F \end{bmatrix}$ as output map if and only if $R = \mathbb{F}^{EC}$ and $S = \mathbb{F}^{BF}$ are input–output maps of well-posed linear systems with $\mathbb{T}$ as semigroup, Φ^E, respectively Φ^B, as input map and Ψ^C, respectively Ψ^F, as output map. ■

From now on, we always suppose the following

Hypothesis 13.
$(A, \mathcal{B}, \mathcal{C})$ is well-posed on $(U^B \times U^E, X, Y_C \times Y_F)$, or equivalently:

- (A, B, C) is well-posed on (U^B, X, Y_C),
- (A, E, F) is well-posed on (U^E, X, Y_F),
- (A, E, C) is well-posed on (U^E, X, Y_C),
- (A, B, F) is well-posed on (U^B, X, Y_F).

Let us denote $\Sigma_\mathcal{A}$ the augmented well-posed linear system on $(U^B \times U^E, X, Y_C \times Y_F)$ whose generating triple is $(A, \mathcal{B}, \mathcal{C})$, then

$$\Sigma_\mathcal{A} = \begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix} = \begin{bmatrix} \mathbb{T} & \begin{bmatrix} \Phi^B & \Phi^E \end{bmatrix} \\ \begin{bmatrix} \Psi^C \\ \Psi^F \end{bmatrix} & \begin{bmatrix} \mathbb{F}^{BC} & \mathbb{F}^{EC} \\ \mathbb{F}^{BF} & \mathbb{F}^{EF} \end{bmatrix} \end{bmatrix}.$$

Let $K \in \mathcal{L}(Y_F, U^E)$ be an admissible feedback operator for Σ_F^E. We close system $\Sigma_\mathcal{A}$ by the feedback law $u^E = Ky_F + v^E$, where v^E is the new control. In other words, let us denote $\mathcal{K} = \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix}$, then define $\Sigma_\mathcal{A}^\mathcal{K}$ the feedback system by (5) with $\Sigma_\mathcal{A}$ instead of Σ and $\mathcal{K}$ instead of K.

Remark 14. As in the finite dimensional situation, this is an easy exercise to show that $\mathcal{K}$ is admissible (for $\Sigma_\mathcal{A}$) if and only if K is admissible (for Σ_F^E).

Then let us denote

$$\Sigma_\mathcal{A}^\mathcal{K} = \begin{bmatrix} \mathbb{T}^K & \Phi^K \\ \Psi^K & \mathbb{F}^K \end{bmatrix} = \begin{bmatrix} \mathbb{T}^K & \begin{bmatrix} \Phi^{B^K} & \Phi^{E^K} \end{bmatrix} \\ \begin{bmatrix} \Psi^{C^K} \\ \Psi^{F^K} \end{bmatrix} & \begin{bmatrix} \mathbb{F}^{B^K C^K} & \mathbb{F}^{E^K C^K} \\ \mathbb{F}^{B^K F^K} & \mathbb{F}^{E^K F^K} \end{bmatrix} \end{bmatrix}.$$

Now we come back to a system with a single input and a single output by taking $v^E \equiv 0$ and by forgetting y_F in Figure 1. We then consider $\Sigma^K = \begin{bmatrix} \mathbb{T}^K & \Phi^{B^K} \\ \Psi^{C^K} & \mathbb{F}^{B^K C^K} \end{bmatrix}$ which is a well-posed linear system, and we have

$$\mathbb{T}^K = \mathbb{T} + \Phi^{E^K} K \Psi^F = \mathbb{T} + \Phi^E K \Psi^{F^K}.$$

This last identity shows that we achieve our goal to construct a well-posed linear system closely related to the initial one, with a semigroup corresponding to the one obtained by closing the loop by K in Σ_F^E.

4. ROBUSTNESS OF EXACT CONTROLLABILITY

Theorem 15. Suppose that Σ is exactly controllable in time $\tau > 0$, that is $\operatorname{Ran} \Phi_\tau^B = X$. Then the following assertions are equivalent:

(1) Σ^K is exactly controllable in time $\tau > 0$, that is $\operatorname{Ran} \Phi_\tau^{B^K} = X$,
(2) $\operatorname{Ran} \Phi_\tau^E \subset \operatorname{Ran} \Phi_\tau^{B^K}$,
(3) $\operatorname{Ran} (\Phi_\tau^B)^* \subset \operatorname{Ran} \left[I + (\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}_\tau^{B^K F^K} \right]$.

Proof.
First note that, from (8) and (9), we have

$$\operatorname{Ran} \Phi_\tau = \operatorname{Ran} \Phi_\tau^K, \quad \operatorname{Ran} \Phi_\tau^{E^K} = \operatorname{Ran} \Phi_\tau^E. \tag{10}$$

"*(1) ⇒ (2)*":
This is obvious.

"*(2) ⇒ (1)*":
Remark that $\operatorname{Ran} \Phi_\tau^B = X$ implies $\operatorname{Ran} \Phi_\tau = X$. Indeed

$$X = \operatorname{Ran} \Phi_\tau^B = \Phi_\tau^B \mathcal{U}^B = \Phi_\tau \left(\mathcal{U}^B \times \{0\} \right) \subset \Phi_\tau \left(\mathcal{U}^B \times \mathcal{U}^E \right) = \operatorname{Ran} \Phi_\tau.$$

Then from (10), $\operatorname{Ran} \Phi_\tau^K = X$. So for all $z_0 \in X$, there exists a pair $(u^B, u^E) \in \mathcal{U}^B \times \mathcal{U}^E$ such that

$$\Phi_\tau^{B^K} u^B + \Phi_\tau^{E^K} u^E = z_0.$$

On the other hand, suppose $\operatorname{Ran} \Phi_\tau^{B^K} \subsetneq X$, *i.e.* that there exists $z_0 \in X$ such that for all $\widetilde{u}^B \in \mathcal{U}^B$, we have $\Phi_\tau^{B^K} \widetilde{u}^B \neq z_0$. This implies from the previous equality that $\exists (u^B, u^E) \in \mathcal{U}^B \times \mathcal{U}^E, \forall \widetilde{u}^B \in \mathcal{U}^B, \Phi_\tau^{B^K} (\widetilde{u}^B - u^B) \neq \Phi_\tau^{E^K} u^E$, which is equivalent to

$$\exists u^E \in \mathcal{U}^E, \forall \overline{u}^B \in \mathcal{U}^B, \quad \Phi_\tau^{B^K} \overline{u}^B \neq \Phi_\tau^{E^K} u^E.$$

This means that

$$\operatorname{Ran} \Phi_\tau^{E^K} \not\subset \operatorname{Ran} \Phi_\tau^{B^K}.$$

We conclude by contraposition, and by using (10) again.

"(1) ⇒ (3)":
Since $\operatorname{Ran} \Phi^B = X$, $\Phi_\tau^B (\Phi_\tau^B)^*$ (the controllability grammian) is invertible in $\mathcal{L}(X)$, so we can rewrite

$$\Phi_\tau^{B^K} = \Phi_\tau^B \left[I + (\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}^{B^K F^K} \right].$$

If $\Phi_\tau^{B^K}$ is surjective, then we must have

$$\left(\operatorname{Ker} \Phi_\tau^B \right)^\perp \subset \operatorname{Ran} \left[I + (\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}^{B^K F^K} \right],$$

where Ker means the kernel of the operator, or equivalently

$$\overline{\operatorname{Ran} (\Phi_\tau^B)^*} \subset \operatorname{Ran} \left[I + (\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}^{B^K F^K} \right].$$

We conclude by claiming that $\operatorname{Ran} (\Phi_\tau^B)^*$ is closed in X. Indeed, since $\operatorname{Ran} \Phi_\tau^B = X$, then $(\Phi_\tau^B)^*$ is bounded from below, hence left-invertible and the result follows from (Brézis, 2011, Theorem 2.13).

"(3) ⇒ (1)":
The range inclusion

$$\operatorname{Ran} (\Phi_\tau^B)^* \subset \operatorname{Ran} \left[I + (\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}_\tau^{B^K F^K} \right],$$

is equivalent to the existence of an $L \in \mathcal{L}(X, \mathcal{U}^B)$ such that

$$(\Phi_\tau^B)^* = \left[I + (\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}_\tau^{B^K F^K} \right] L.$$

If we multiply by Φ_τ^B on the left, we get

$$\exists L \in \mathcal{L}(X, \mathcal{U}^B), \quad \Phi_\tau^B (\Phi_\tau^B)^* = \Phi_\tau^{B^K} L.$$

So that

$$X = \operatorname{Ran} \Phi_\tau^B (\Phi_\tau^B)^* \subset \operatorname{Ran} \Phi_\tau^{B^K}.$$

■

Corollary 16. Suppose that Σ is exactly controllable in time $\tau > 0$. There exists $\vartheta > 0$ such that for all $K \in \mathcal{L}(Y, U)$ with $\|K\| < \vartheta$, Σ^K is exactly controllable in time $\tau > 0$.

Proof. From Proposition 10, there exists $d > 0$ (possibly $= \infty$) such that all $K \in \mathcal{L}(Y_F, U^E)$ with $\|K\| < d$ are admissible output feedback operators for Σ_F^E. If furthermore, $\|K\| < \|\mathbb{F}_\tau^{EF}\|^{-1}$, we then have

$$\|(I - \mathbb{F}_\tau^{EF} K)^{-1}\| \leq \frac{1}{1 - \|\mathbb{F}_\tau^{EF}\| \|K\|}. \tag{11}$$

Now, if

$$\|(\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}_\tau^{B^K F^K} \| < 1,$$

Then $I + (\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}_\tau^{B^K F^K}$ is invertible, so its range is the whole space $\mathcal{U}^B$ and point *(3)* of Theorem 15 is satisfied.

We have

$$\begin{aligned} &\|(\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}_\tau^{B^K F^K} \| \\ &\qquad \leq \|(\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E \| \|K\| \|\mathbb{F}_\tau^{B^K F^K}\|. \end{aligned}$$

And from (5) and (7), we know that

$$\mathbb{F}_\tau^{B^K F^K} = (I - \mathbb{F}_\tau^{EF} K)^{-1} \mathbb{F}_\tau^{BF},$$

so we have thanks to (11)

$$\begin{aligned} &\|(\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E K \mathbb{F}_\tau^{B^K F^K} \| \\ &\quad \leq \|(\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E \| \frac{\|K\|}{1 - \|\mathbb{F}_\tau^{EF}\| \|K\|} \|\mathbb{F}_\tau^{BF}\|. \end{aligned}$$

After straightforward computations, we show that

$$\|(\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E \| \frac{\|K\|}{1 - \|\mathbb{F}_\tau^{EF}\| \|K\|} \|\mathbb{F}_\tau^{BF}\| < 1,$$

if and only if

$$\|K\| < \left[\|(\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E \| \|\mathbb{F}_\tau^{BF}\| + \|\mathbb{F}_\tau^{EF}\|^{-1} \right]^{-1},$$

which together with the two other bounds for K gives that ϑ is given by

$$\min \left\{ d, \|\mathbb{F}_\tau^{EF}\|^{-1}, \left[\|(\Phi_\tau^B)^* \left[\Phi_\tau^B (\Phi_\tau^B)^* \right]^{-1} \Phi_\tau^E \| \|\mathbb{F}_\tau^{BF}\| + \|\mathbb{F}_\tau^{EF}\|^{-1} \right]^{-1} \right\},$$

and the result follows. ■

Remark 17. Our Theorem 15 can be thought to be very theoretic and not usable in practice since it is very difficult to characterise the range of an operator in general. However, it allows us to avoid the description of Banach spaces involving unbounded operators (namely admissible ones), as in (Hadd, 2005, Definition 2.2), to prove Corollary 16. Furthermore, we can tackle more general perturbations with this approach. Corollary 16 is the main result of this work, showing that, under Hypothesis 13, for all bounded operator K small enough, exact controllability persists.

5. THE REGULAR CASE

Definition 18. Let Σ be well-posed linear system on (U, X, Y). If for all $u \in U$, the limit

$$\lim_{\mathbb{R} \ni \lambda \to \infty} \mathbf{G}(\lambda) u,$$

exists, then Σ is said to be *regular*. The limit is denoted Du and allows to define $D \in \mathcal{L}(U, Y)$ the *feedthrough operator* of Σ.

Remark 19. Regular linear systems admit various equivalent definitions, see Tucsnak and Weiss (2014).

Remark 20. As said in (Hadd, 2005, Remark 2.5), if the control or observation operator is bounded, then the resulting triple is a generating triple and the system is regular with $D = 0$ as feedthrough operator.

In Hadd (2005); Mei and Peng (2010), they suppose that Σ is a regular linear system (more precisely, they consider control systems, so that $C = 0$ and then regularity follows from the previous remark). They consider perturbations given by admissible control or observation operators. In the following, we show that our result, namely Corollary 16, contains their results. Let $(A, B, 0)$ be the generating triple of Σ, with 0 as feedthrough operator.

In Hadd (2005), the author consider "control perturbation" P^B, that is an admissible control operator. Define $\mathcal{B} = \begin{bmatrix} B & P^B \end{bmatrix}$ and $\mathcal{C} = \begin{bmatrix} 0 \\ I \end{bmatrix}$. Then, from the previous remark, we get that Hypothesis 13 is satisfied, and that the four systems are regular with feedthrough operators 0. Finally, from (Tucsnak and Weiss, 2014, Theorem 5.17) and Corollary 16, we get that $A^K = (A + P^B K)$ on its domain, $B^K = B$, and the result in (Hadd, 2005, Theorem 3.3-(iii)) with smallness assumption needed on the operator K.

In Mei and Peng (2010), the authors consider "observation perturbation" P^O (for the controllability problem). They suppose that (A, B, P^O) is the generating triple of a regular linear system. Furthermore, it seems that they implicitly assume that the feedthrough operator of the regular linear system is $D = 0$. Indeed, in Theorem 2.2, they reproduce the results on regular linear systems with feedback of Weiss (1994a), summed up in (Tucsnak and Weiss, 2014, Theorem 5.17). However, a comparison of the two above theorems show, for instance, that $(I - DK)^{-1} = I$, by getting a close look at the form of A^K. Furthermore, it is said in the proof of Theorem 3.9 that the transfer function associated with (A, B, P^O) tends to zero at infinity, which means by definition that $D = 0$. These two facts steer us to believe that the authors have made this assumption implicitly. So let us suppose that (A, B, P^O) is the generating triple of a regular linear system with feedthrough operator $D = 0$. Define $\mathcal{B} = [B \ I]$ and $\mathcal{C} = \begin{bmatrix} 0 \\ P^O \end{bmatrix}$. Then again, Hypothesis 13 is satisfied and the four systems are regular with feedthrough operators 0. And again from (Tucsnak and Weiss, 2014, Theorem 5.17) and Corollary 16 we get that $A^K = (A + KP^O_\Lambda)$ on its domain, $B^K = B$, and the result in (Mei and Peng, 2010, Theorem 3.9) with smallness assumption needed on the operator K. Note that we use here P^O_Λ, the Λ-extension of the observation operator P^O, see (Tucsnak and Weiss, 2014, Definition 5.1).

6. CONCLUSION

As a conclusion, we can say that the idea of "extended well-posed linear system" introduced in Hadd (2005) allows to consider more general perturbations in Corollary 16 for controllable systems than just admissible ones, by seeing them as feedback laws. However, we can not conclude without mentioning again that in general, we also perturb the way we control and the way we observe (*i.e.* the control and the observation operators), which can be a major drawback. Indeed, if we want, for instance, study control and/or observation of a perturbation of a linear partial differential equation (PDE) in the context of WPLS, the control and observation operators of the unperturbed PDE have a physical meaning: we do not know if we can give to the perturbed control and observation operators a relevant physical meaning for the target perturbed PDE.

We also mention that the dual counterpart of the present results, namely exact observability, will lead to a generalization of the other result of Mei and Peng (2010): considering "control perturbation" for observation systems.

REFERENCES

Boulite, S., Idrissi, A., and Maniar, L. (2005). Robustness of controllability under some unbounded perturbations. *J. Math. Anal. Appl.*, 304(1), 409–421.

Brézis, H. (2011). *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York.

Cîndea, N. and Tucsnak, M. (2010). Internal exact observability of a perturbed euler-bernoulli equation. *Ann. Acad. Rom. Sci. Ser. Math. its Appl.*, 2(2), 205–221.

Curtain, R.F. and Weiss, G. (1989). Well posedness of triples of operators (in the sense of linear systems theory). In *Control Estim. Distrib. Param. Syst. (Vorau, 1988)*, volume 91, 41–59. Birkhuser.

Hadd, S. (2005). Exact controllability of infinite dimensional systems persists under small perturbations. *J. Evol. Equations*, 5(4), 545–555.

Lee, E. and Markus, L. (1967). *Foundations of optimal control theory.* SIAM series in applied mathematics. Wiley.

Leiva, H. (2003). Unbounded perturbation of the controllability for evolution equations. *J. Math. Anal. Appl.*, 280(1), 1–8.

Mei, Z.d. and Peng, J.g. (2010). On robustness of exact controllability and exact observability under cross perturbations of the generator in banach spaces. *Proc. Am. Math. Soc.*, 138(12), 4455–4468.

Mei, Z.D. and Peng, J.G. (2014). Robustness of exact p-controllability and exact p-observability to q-type of perturbations of the generator. *Asian J. Control*, 16(4), 1164–1168.

Opmeer, M.R. (2005). Infinite-dimensional linear systems: a distributional approach. *Proc. London Math. Soc. (3)*, 91(3), 738–760.

Salamon, D. (1987). Infinite dimensional linear systems with undbounded bontrol and observation: A functional analytic approach. *Trans. Am. Math. Soc.*, 300(2), 383–431.

Salamon, D. (1989). Realization theory in Hilbert space. *Math. Syst. Theory*, 21(1), 147–164.

Staffans, O. (2005). *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge.

Tucsnak, M. and Weiss, G. (2009). *Observation and control for operator semigroups.* Birkhäuser Advanced Texts: Basel Textbooks. Birkhäuser Verlag, Basel.

Tucsnak, M. and Weiss, G. (2014). Well-posed systems - The LTI case and beyond. *Automatica*, 50(7), 1757–1779.

Weiss, G. (1989a). Admissible observation operators for linear semigroups. *Isr. J. Math.*, 65(1), 17–43.

Weiss, G. (1989b). The representation of regular linear systems on Hilbert spaces. In *Control Estim. Distrib. Param. Syst. (Vorau, 1988)*, volume 91, 401—-416. Birkhuser.

Weiss, G. (1994a). Regular linear systems with feedback. *Math. Control. Signals, Syst.*, 7(1), 23–57.

Weiss, G. (1994b). Transfer Functions of Regular Linear Systems. Part I: Characterizations of Regularity. *Trans. Am. Math. Soc.*, 342(2), 827–854.

Weiss, G., Staffans, O.J., and Tucsnak, M. (2001). Well-posed linear systems - a survey with emphasis on conservative systems. *Int. J. Appl. Math. Comput. Sci.*, 11(1), 7–33.