Recovering the observable part of the initial data of an infinite-dimensional linear system with perturbed skew-adjoint generator using observers

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***Abstract*— We consider the problem of recovering the initial data (or initial state) of infinite-dimensional linear systems generated by a perturbed skew-adjoint operator. It is well-known that this inverse problem is well posed if the system is exactly observable, but this assumption may be very restrictive in some applications. In this paper we are interested in the slight generalization of the results by Haine [12] in the case where the generator is a simple perturbation of a skew-adjoint operator. The reconstruction algorithm is based on iterative forward and backward observers, using the algorithm of Ramdani, Tucsnak and Weiss [19].**

I. INTRODUCTION

In many areas of science, we need to recover the initial (or final) data of a physical system from partial observations, sometimes over a finite time interval. In the last decade, new algorithms based on time reversal have been proposed (see Fink [7], [8]). We can mention the Back and Forth Nudging proposed by Auroux and Blum [1], the Time Reversal Focusing by Phung and Zhang [18] and finally, the one we will consider in this work, the forward–backward observer-based algorithm proposed by Ramdani, Tucsnak and Weiss [19]. In this paper we study the convergence of the reconstruction algorithm of [19] for systems with *perturbation of skew-adjoint* generator, when the inverse problem is ill-posed, that is to say when either the observability or the estimatability assumption fails.

In order to make this statement precise, let us begin with some notation and definitions.

Let X be a Hilbert space and $\mathcal{A} \in \mathcal{L}(\mathcal{D}(\mathcal{A}), X)$ a skew-adjoint operator. By Stone's theorem, it generates a unitary C_0-group $\mathbb{S}$. For $\alpha \in \mathbb{R}$, we consider the bounded perturbation $A = \mathcal{A} + \alpha I$, where I is the identity of $\mathcal{L}(X)$, with domain $\mathcal{D}(A) = \mathcal{D}(\mathcal{A})$. Note that this implies in particular that A is the generator of a C_0-group $\mathbb{T}$. Furthermore, we have $\mathbb{T}_t = \mathbb{S}_t e^{\alpha t} = e^{\alpha t}\mathbb{S}_t$ for all $t \in \mathbb{R}$.

We are interested in the reconstruction of the initial data z_0 of

$$\begin{cases} \dot{z}(t) = Az(t) & \forall\, t \geq 0, \\ z(0) = z_0 \in X. \end{cases} \tag{1}$$

Such equations are often used to model damped vibrating systems (acoustic or elastic waves) or quantum systems (Schrödinger equations).

Let Y be another Hilbert space. We suppose that we have access to z through the operator $C : \mathcal{D}(A) \to Y$, during a time interval $[0, \tau]$, $\tau > 0$, leading to the measurement

$$y(t) = Cz(t) \qquad \forall\, t \in [0, \tau]. \tag{2}$$

We call C the observation operator of the system. The observation is said to be *bounded* if C is a bounded operator (*i.e.* $C \in \mathcal{L}(X, Y)$), and unbounded otherwise. In the latter case, we still assume that C is bounded with respect to the graph norm of A on $\mathcal{D}(A)$.

For systems described by evolution partial differential equations (*i.e.* when A is a differential operator in the space variables on a domain Ω), bounded observation generally corresponds to measurement on a subdomain $\mathcal{O} \subset \Omega$, while unbounded observation in most cases corresponds to measurement on the boundary of Ω.

If we denote Ψ_τ the operator which associates the output function $y|_{[0,\tau]}$ to an initial data $z_0 \in \mathcal{D}(A)$, the inverse problem is well-posed when Ψ_τ is left-invertible, with bounded left-inverse. This is equivalent to Ψ_τ being bounded from below, *i.e.*

$$\exists k_\tau > 0, \quad \|\Psi_\tau z_0\| \geq k_\tau \|z_0\| \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{3}$$

The pair (A, C) is said to be *exactly observable in time* τ when (3) holds.

Now, we present the algorithm proposed by Ramdani, Tucsnak and Weiss [19]. For simplicity, we consider the particular case where A is skew-adjoint and $C \in \mathcal{L}(X, Y)$, the pair (A, C) being exactly observable in time $\tau > 0$. Let $\mathbb{T}^+$ be the exponentially stable C_0-semigroup generated by $A^+ = A - \gamma C^* C$, while $\mathbb{T}^-$ is generated by $A^- = -A - \gamma C^* C$, for some $\gamma > 0$ (see Liu [16]). For all $n \in \mathbb{N} \setminus \{0\}$, we define the following systems

$$\begin{cases} \dot{z}_n^+(t) = A^+ z_n^+(t) + \gamma C^* y(t) \quad \forall t \in [0, \tau], \\ z_1^+(0) = z_0^+ \in X, \\ z_n^+(0) = z_{n-1}^-(0) \quad \forall n \geq 2, \end{cases} \tag{4}$$

$$\begin{cases} \dot{z}_n^-(t) = -A^- z_n^-(t) - \gamma C^* y(t) \quad \forall t \in [0, \tau], \\ z_n^-(\tau) = z_n^+(\tau) \quad \forall n \geq 1. \end{cases} \tag{5}$$

The forward error $e_n^+(t) = z_n^+(t) - z(t)$ satisfies

$$\begin{cases} \dot{e}_n^+(t) = (A - \gamma C^* C) e_n^+(t) \quad \forall t \in [0, \tau], \\ e_1^+(0) = z_0^+ - z_0 \in X, \\ e_n^+(0) = e_{n-1}^-(0) \quad \forall n \geq 2, \end{cases}$$

and the backward error $e_n^-(t) = z_n^-(t) - z(t)$

$$\begin{cases} \dot{e}_n^-(t) = (A + \gamma C^* C) e_n^-(t) \quad \forall t \in [0, \tau], \\ e_n^-(\tau) = e_n^+(\tau) \quad \forall n \geq 1. \end{cases}$$

So, we have

$$\begin{aligned} \|z_n^-(0) - z_0\| &= \|e_n^-(0)\| \\ &= \left\| \left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)^n e_1^+(0) \right\| \\ &\leq \left\| \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \right\|^n \left\| z_0^+ - z_0 \right\|. \end{aligned} \tag{6}$$

According to Ito, Ramdani and Tucsnak [14, Lemma 2.2], if (A, C) is exactly observable in time τ, we have $r = \|\mathbb{T}_\tau^- \mathbb{T}_\tau^+\|_{\mathcal{L}(X)} < 1$ and thus

$$\|z_n^-(0) - z_0\| \leq r^n \|z_0^+ - z_0\| \underset{n \to \infty}{\longrightarrow} 0.$$

In the case of exactly observable systems, we call the systems (4)–(5) *forward* and *backward observers* as it is a generalization to infinite dimensional systems of the so-called Luenberger's observers [17], well-known in control theory. Observers for infinite dimensional systems are an active topic of research, for both linear or non-linear systems. In the last decade, a lot of papers was published, and among this large literarture, we can mention the recent work of Bertoglio, Chapelle, Fernandez, Gerbeau and Moireau [4], Fridman [9], Krstic, Guo and Smyshlyaev [15]. For pioneering work on the generalisation of observers to infinite dimensional systems, we refer to Baras and Bensoussan [2] and Bensoussan [3].

In the paper of Ramdani, Tucsnak and Weiss [19], they consider a wide class of infinite-dimensional systems (allowing even an observation operator that is not admissible). They suppose that the system is estimatable and backward estimatable (roughly speaking, the system can be forward and backward stabilized with a feedback operator called a *stabilizing output injection operator*). However, they show in Proposition 3.3 that this implies that the system is exactly observable, or in other words, that (3) is satisfied (for some sufficiently large time τ). In this paper we are dealing with the initial data recovery of some well-posed linear systems which are not supposed to be exactly observable, using the same algorithm.

II. Functional framework

Let X, and Y be two Hilbert spaces. We consider a well-posed linear system $\Sigma = \begin{bmatrix} \mathbb{T}_t & \Phi_t \\ \Psi_t & \mathbb{F}_t \end{bmatrix}$, generated by the triple (A, C^*, C), with state space X, and control and observation space Y. We refer to Weiss, Staffans and Tucsnak [22] and Tucsnak and Weiss [20] and the references therein for more details on well-posed linear systems.

We suppose in the sequel that $\mathbb{T}$ is generated by an infinitesimal generator $A = \mathcal{A} + \alpha I$, for some $\alpha \in \mathbb{R}$ and skew-adjoint operator $\mathcal{A}$.

A. The dual system

Let us begin with the definition of the *time-reflection operator*. Let W be a Hilbert space. For all $\tau \geq 0$, we define the linear operator $\text{Я}_\tau : L^2_{\ell oc}([0, \infty), W) \to L^2_{\ell oc}([0, \infty), W)$ by

$$(\text{Я}_\tau u)(t) = \begin{cases} u(\tau - t) & \forall\, t \in [0, \tau], \\ 0 & \forall\, t > \tau. \end{cases}$$

Theorem 2.1 (Theorem 4 of [22]): Let $\Sigma = \begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix}$ be a well-posed linear system with input space U, state space X and output space Y. Define $\Sigma^d = \left(\Sigma_t^d\right)_{t \geq 0}$ by

$$\begin{bmatrix} \mathbb{T}_t^d & \Phi_t^d \\ \Psi_t^d & \mathbb{F}_t^d \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \text{Я}_t \end{bmatrix} \begin{bmatrix} \mathbb{T}_t^* & \Psi_t^* \\ \Phi_t^* & \mathbb{F}_t^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \text{Я}_t \end{bmatrix}. \tag{7}$$

Then $\Sigma^d = \begin{bmatrix} \mathbb{T}^d & \Phi^d \\ \Psi^d & \mathbb{F}^d \end{bmatrix}$ is a well-posed linear system with input space Y, state space X and output space U.

In particular, $\omega_0(\mathbb{T}) = \omega_0(\mathbb{T}^d)$. The linear system Σ^d is called the dual system of Σ.

Proposition 2.2 (Proposition 4 of [22]): If A, B and C are respectively the semigroup generator, control operator and observation operator of the system Σ with growth bound $\omega_0(\mathbb{T})$, then the corresponding operators for Σ^d are A^*, C^* and B^*.

B. Feedback law

Theorem 2.3 (Corollary of Theorem 5.8 [5]): Suppose that Σ is a well-posed linear system such that $A = \mathcal{A} + \alpha I$, for some $\alpha \in \mathbb{R}$ and skew-adjoint operator $\mathcal{A}$. Then there exists a $\kappa > 0$ (possibly $\kappa = +\infty$) such that for all $\gamma \in (0, \kappa)$, the feedback law $-\gamma y + v$ (v is the new control) leads to a closed-loop system Σ^γ which is well-posed.

Remark 2.1: The parameter γ is called the gain, and we can tune it to modify the growth bound of the closed-loop system.

III. Construction of the observers

Let $\Sigma^{\mathcal{A}}$ be the well-posed linear system with skew-adjoint generator $\mathcal{A}$. As in [12], we construct two well-posed linear systems $\widetilde{\Sigma^+} = \begin{bmatrix} \mathbb{S}^+ & \widetilde{\Phi}^+ \\ \widetilde{\Psi}^+ & \widetilde{\mathbb{F}}^+ \end{bmatrix}$ and $\widetilde{\Sigma^-} = \begin{bmatrix} \mathbb{S}^- & \widetilde{\Phi}^- \\ \widetilde{\Psi}^- & \widetilde{\mathbb{F}}^- \end{bmatrix}$ as the closed-loop system of $\Sigma^{\mathcal{A}}$ and $\left(\Sigma^{\mathcal{A}}\right)^d$ respectively, with the same parameter γ (see [12, Lemma 3.1]).

We denote by $(\mathcal{A}^+, C^*, C)$ and $(\mathcal{A}^-, C^*, C)$ the generating triples of $\widetilde{\Sigma^+}$ and $\widetilde{\Sigma^-}$ respectively. Then, we define Σ^+ as the perturbation of $\mathcal{A}^+$ by αI and Σ^- as the perturbation of $\mathcal{A}^-$ by $-\alpha I$. We denote $A^+ = \mathcal{A}^+ + \alpha I$ the infinitesimal generator of the C_0-group $\mathbb{T}^+$ of

$$\Sigma^+ = \begin{bmatrix} \mathbb{T}^+ & \Phi^+ \\ \Psi^+ & \mathbb{F}^+ \end{bmatrix}, \tag{8}$$

and $A^- = \mathcal{A}^- - \alpha I$ the infinitesimal generator of the C_0-group $\mathbb{T}^-$ of

$$\Sigma^- = \begin{bmatrix} \mathbb{T}^- & \Phi^- \\ \Psi^- & \mathbb{F}^- \end{bmatrix}. \tag{9}$$

Remark 3.1: It can be shown that (8) and (9) correspond in fact to the closed-loop system, with feedback operator γI, of Σ and $(\Sigma)^d$ respectively, the systems respectively generated by (A, C^*, C) and (A^*, C^*, C). In other words, we can either construct the observers for the unperturbed system and add the perturbation, as we did it, or construct directly the observers from the perturbed system, this will lead to the same systems Σ^+ and Σ^-.

IV. Main result

Let us introduce the following orthogonal decomposition of an element z of X.

Lemma 4.1: With the previous notation and definitions, we have

$$X = \operatorname{Ker} \Psi_\tau \oplus \overline{\operatorname{Ran} \Phi_\tau^d}.$$

Proof: See [12]. ∎

We now state the main result of this paper.

Theorem 4.2: Let X and Y be Hilbert spaces. Assume that Σ is a well-posed linear system such that $A = \mathcal{A} + \alpha I$, for some $\alpha \in \mathbb{R}$ and skew-adjoint operator $\mathcal{A}$. Let us denote by Σ^+ and Σ^- the systems defined by (8) and (9) respectively.

Let $z_0 \in X$ and denote u, z and y the input, trajectory and output of Σ respectively, with initial state z_0. Let $\tau > 0$, $z_0^+ \in V_{\text{Obs}} = \overline{\operatorname{Ran} \Phi_\tau^d}$ and denote, for all $n \geq 1$, z_n^+ and z_n^- the respective trajectories of Σ^+ and Σ^- with respective inputs $v^+ = \gamma y + u$ and $v^- = \gamma Я_\tau y + Я_\tau u$, and initial states

$$z_1^+(0) = z_0^+ \in X, \quad z_n^+(0) = z_{n-1}^-(0), \ n \geq 2,$$
$$z_n^-(\tau) = z_n^+(\tau), \ n \geq 1.$$

Furthermore, we denote by Π the orthogonal projector from X onto V_{Obs}, then the following statements hold true:

1) We have for all $z_0 \in X, z_0^+ \in V_{\text{Obs}}$ and $n \geq 1$
$$(I - \Pi)\left(z_0 - z_n^-(0)\right) = (I - \Pi)\, z_0.$$
2) The sequence $(\|z_n^-(0) - \Pi z_0\|)_{n \geq 1}$ is strictly decreasing and satisfies
$$\left\|z_n^-(0) - \Pi z_0\right\| = o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.$$
3) The rate of convergence is exponential, *i.e.* there exists a constant $r \in (0, 1)$, independent of z_0 and z_0^+, such that for all $n \geq 1$
$$\left\|z_n^-(0) - \Pi z_0\right\| \leq r^n \left\|\Pi z_0\right\|,$$
if and only if $\operatorname{Ran} \Phi_\tau^d$ is closed in X.

V. Proof

To prove Theorem 4.2, we first have to show that

$$z_n^-(0) - z_0 = \left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)^n \left(z_0^+ - z_0\right) \quad \forall n \geq 1, \tag{10}$$

and then study the *forward–backward operator* $\mathbb{T}_\tau^- \mathbb{T}_\tau^+ \in \mathcal{L}(X)$.

Remark that once we have proved (10), everything is done in [12], [11]. Indeed, we have $\mathbb{T}_t^+ = e^{\alpha t}\mathbb{S}_t^+ = \mathbb{S}_t^+ e^{\alpha t}$ and $\mathbb{T}_t^- = e^{-\alpha t}\mathbb{S}_t^- = \mathbb{S}_t^- e^{-\alpha t}$ for all $t \in \mathbb{R}$. Thus $\mathbb{T}_\tau^-\mathbb{T}_\tau^+ = \mathbb{S}_\tau^-\mathbb{S}_\tau^+$ and the results of [12], [11] apply.

For all $z_0^+, z_0 \in X$, $t \in (0,\tau)$, we have

$$z(t) = \mathbb{T}_t z_0, \quad z^+(t) = z_1^+(t) = \mathbb{T}_t^+ z_0^+ + \gamma\Phi_t^+ y,$$

and

$$z^-(t) = z_1^-(t) = \mathbb{T}_t^- z^+(\tau) + \gamma\Phi_t^- \text{Я}_\tau y.$$

From straightforward computations on the link between

$$\widetilde{\Sigma^+} = \begin{bmatrix} \mathbb{S}^+ & \widetilde{\Phi}^+ \\ \widetilde{\Psi}^+ & \widetilde{\mathbb{F}}^+ \end{bmatrix} \text{ and } \Sigma^+ = \begin{bmatrix} \mathbb{T}^+ & \Phi^+ \\ \Psi^+ & \mathbb{F}^+ \end{bmatrix},$$

and between

$$\widetilde{\Sigma^-} = \begin{bmatrix} \mathbb{S}^- & \widetilde{\Phi}^- \\ \widetilde{\Psi}^- & \widetilde{\mathbb{F}}^- \end{bmatrix} \text{ and } \Sigma^- = \begin{bmatrix} \mathbb{T}^- & \Phi^- \\ \Psi^- & \mathbb{F}^- \end{bmatrix},$$

we can easily show that

$$z^+(\tau) - z(\tau) = \mathbb{T}_\tau^+\left(z_0^+ - z_0\right) \quad \forall z_0^+, z_0 \in X,$$

and

$$\text{Я}_\tau z^-(\tau) - z(0) = \mathbb{T}_\tau^-\mathbb{T}_\tau^+\left(z_0^+ - z_0\right) \quad \forall z_0^+, z_0 \in X.$$

Whence the results.

VI. Numerical simulations

Let us consider the toy model of Schrödinger equations in one dimension with real constant potential and homogeneous Dirichlet boundary condition. In other words, we are interested in

$$\begin{cases} \dfrac{\partial}{\partial t} z = -\mathbf{i}\dfrac{\partial^2}{\partial x^2} z + \alpha z & \forall x \in (0,1), t \geq 0, \\ z(t,0) = z(t,1) = 0 & \forall t \geq 0, \\ z(0,x) = z_0(x) & \forall x \in (0,1), \end{cases}$$

for some real parameter α, *the constant potential*.

We suppose we have access to three different observations on the subinterval $(0, 0.1)$ during a time $\tau = 0.2$, given by

$$\begin{cases} y_1(t,x) = z(t,x)|_{x\in(0,0.1)} & \forall t \in (0,0.2), \\ y_2(t,x) = \operatorname{Re} z(t,x)|_{x\in(0,0.1)} & \forall t \in (0,0.2), \\ y_3(t,x) = \mathbf{i}\operatorname{Im} z(t,x)|_{x\in(0,0.1)} & \forall t \in (0,0.2). \end{cases}$$

It is well-known that with y_1, $\alpha = 0$, the system is exactly observable and thus we will be able to reconstruct the whole initial data. In the case y_2 and y_3, with $\alpha = 0$, the results of [12] apply. In the particular example we are looking at, the observation operator is bounded and self-adjoint and thus we can apply the result of Liu [16] to define very easily the feedback law.

The algorithm takes the following form in terms of partial differential Schrödinger equations. For all $n \in \mathbb{N}$, $k = 1,2,3$, the forward observer is

$$\begin{cases} \dfrac{\partial}{\partial t} z_n^+ = -\mathbf{i}\dfrac{\partial^2}{\partial x^2} z_n^+ + \alpha z_n^+ & \\ \qquad -\gamma\chi z_n^+ + \gamma y_k & \forall x \in (0,1), t \geq 0, \\ z_n^+(t,0) = z_n^+(t,1) = 0 & \forall t \geq 0, \\ z_n^+(0,x) = z_{n-1}^-(\tau,x) & \forall x \in (0,1), n \geq 1, \\ z_1^+(0,x) = 0 & \forall x \in (0,1), \end{cases}$$

and the backward observer is

$$\begin{cases} \dfrac{\partial}{\partial t} z_n^- = \mathbf{i}\dfrac{\partial^2}{\partial x^2} z_n^- - \alpha z_n^- & \\ \qquad +\gamma\chi z_n^- - \gamma \text{Я}_\tau y_k & \forall x \in (0,1), t \geq 0, \\ z_n^-(t,0) = z_n^-(t,1) = 0 & \forall t \geq 0, \\ z_n^-(0,x) = z_n^+(\tau,x) & \forall x \in (0,1), n \geq 0, \end{cases}$$

where χ is the characteristic function of the subinterval $(0, 0.1)$ and γ is the gain parameter.

Remark that from $\mathbb{T}_\tau^-\mathbb{T}_\tau^+ = \mathbb{S}_\tau^-\mathbb{S}_\tau^+$, we can expect that the perturbation will not induce any modification on the error plots, provided that the mesh parameters are suitably choosen to ensure that $e^{\pm\alpha\tau}$ is sufficiently well approximated.

We perform some tests with a Gaussian noise (standard deviation equals 2) on the measurement. The discretization is made using finite element of order one in space and first order central difference in time. The initial data we want to recover, with a gain parameter $\gamma = 50$, is the following

$$z_0(x) = 100\cos(5\pi x)\sin(\pi x)e^{-50(x-0.325)^2} + \mathbf{i}30\cos(7.5\pi x)\sin(0.5\pi x)e^{-50(x-0.75)^2}.$$

On Fig. 1, we can observe that the error plot for $\alpha = 0$, $\alpha = -15$ and $\alpha = 15$ are exactly the same, as expected.

In the three cases, we obtain the following reconstruction of the initial data. As we can see on Fig. 2, we reconstruct the observable part of the initial data with observation y_2 and y_3 (the systems are not exactly observable). Furthermore, note that these two observations y_2 and y_3 are complementary, *i.e.* $y_1 = y_2 + y_3$. Thus, intuitively, we can hope that using both partial reconstructions, we will find the whole initial data, and indeed, when we add the two partial reconstructions we find exactly the one obtained in the exactly observable case.

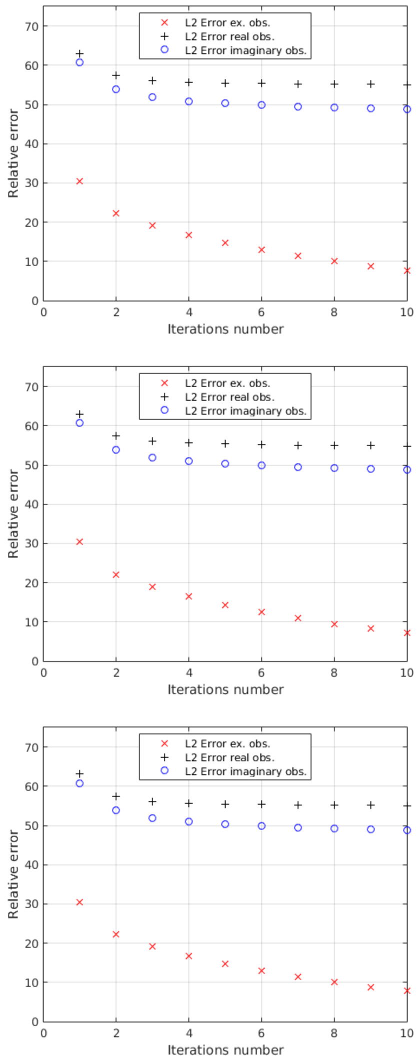

Fig. 1: Comparison of the error plots with $\alpha = 0, -15$ and 15.

VII. CONJECTURE

In this section, we numerically test the algorithm on the following Schrödinger's equation

$$\begin{cases} \dfrac{\partial}{\partial t} z = -\mathbf{i}\dfrac{\partial^2}{\partial x^2} z + \alpha\theta z & \forall x \in (0,1), t \geq 0, \\ z(t,0) = z(t,1) = 0 & \forall t \geq 0, \\ z(0,x) = z_0(x) & \forall x \in (0,1), \end{cases}$$

where θ is the characteristic function of the subinterval $(0.75, 1)$. This locally distributed perturbation does not fit into the framework of Theorem 4.2.

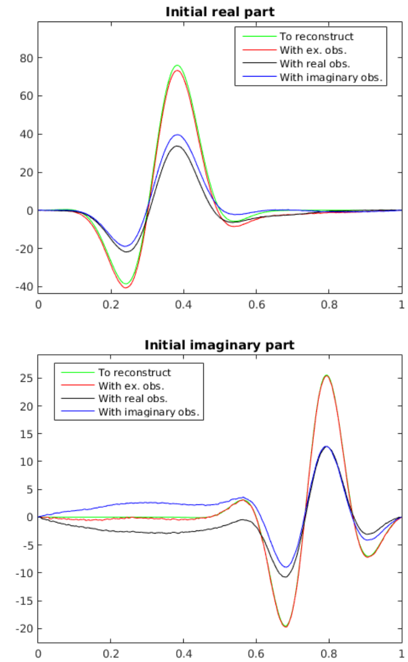

Fig. 2: The reconstruction obtained with the three kind of observations (identical in the three cases $\alpha = 0, -15$ and 15).

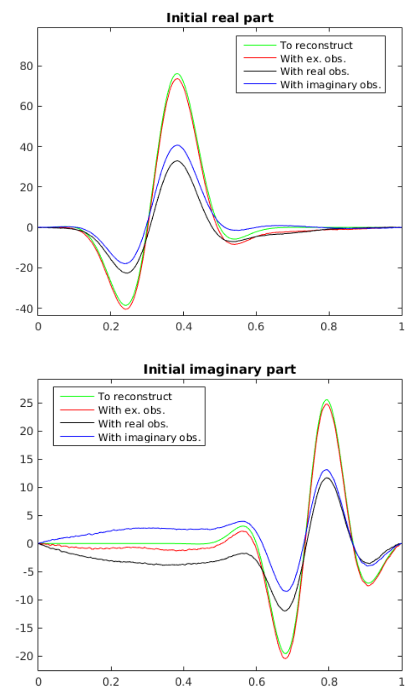

Fig. 3: The reconstruction obtained with locally distributed perturbation on $(0.75, 1)$.

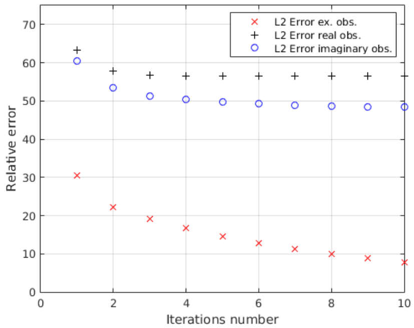

Fig. 4: Error plots with $\alpha = 15$ with locally distributed perturbation on $(0.75, 1)$.

We can observe on Fig. 3 and Fig. 4 that the algorithm seems to be robust to this perturbation. This lead us to state the folowing conjecture.

Conjecture 7.1: Let X and Y be Hilbert spaces. Assume that Σ is a well-posed linear system such that $A = \mathcal{A} + P$, for some $P \in \mathcal{L}(X)$ and skew-adjoint operator $\mathcal{A}$. Then the conclusions of Theorem 4.2 holds.

VIII. Conclusion

In conclusion, we propose a slight generalization of the results obtained in [12], [11], by adding a simple perturbation of the kind αI, with $\alpha \in \mathbb{R}$, to the generator of the C_0-group of the considered system. Our goal, though, is to add an arbitrary bounded perturbation an not only αI: it is still ongoing work.

Note that the noise added in the simulation is not taken into account in the proofs. In particular, we think that the decomposition of the state space X is not fully preserved in presence of noise. However, in practice, it seems that the deterioration of the reconstruction is negligible. We also mention that there exists an optimal number of iterations depending on the mesh parameters when we use *simple* discretization process, as we can read in Haine and Ramdani [13]. Using a numerical viscosity method, as in the paper of Ervedoza and Zuazua [6], we can remove this limitation of the approach by the algorithm of [19]. It has been done successfully in a recent work of Garcìa and Takahashi [10].

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