RECONSTRUCTING INITIAL DATA USING ITERATIVE OBSERVERS FOR WAVE TYPE SYSTEMS. A NUMERICAL ANALYSIS

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Talk Abstract

A new iterative algorithm for solving initial data inverse problems from partial observations has been recently proposed in Ramdani, Tucsnak and Weiss [1]. In this work, we are concerned with the convergence analysis of this algorithm. We provide a complete numerical analysis for a fully discrete approximation derived using finite elements in space and finite differences in time. We present these results in the case of wave conservative systems with locally distributed observation and conclude with a numerical example.

Introduction

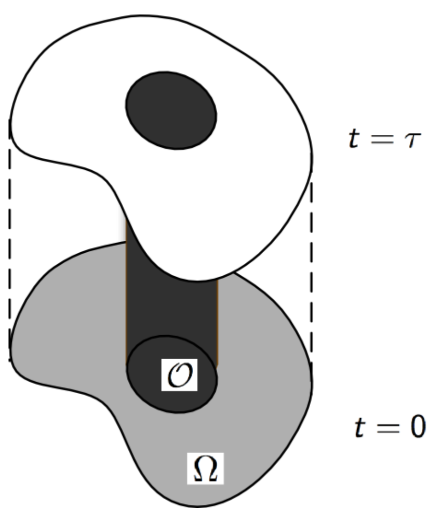

Figure 1: An initial data inverse problem for evolution PDE's : How to reconstruct the initial state (light grey) for a PDE set on a domain Ω from partial observation on $\mathcal{O} \times [0, \tau]$ (dark grey)?

Let us start by briefly recalling the principle of the reconstruction method proposed in [1] in the simplified context of skew-adjoint generators and bounded observation operator. We will always work under these assumptions throughout the paper. Given two Hilbert spaces X and Y (called *state* and *output* spaces respectively), let $A : \mathcal{D}(A) \to X$ be skew-adjoint operator generating a C_0-group $\mathbb{T}$ of isometries on X and let $C \in \mathcal{L}(X, Y)$ be a bounded observation operator. Consider the infinite dimensional linear system given by

$$\begin{cases} \dot{z}(t) = Az(t), & \forall t \geq 0, \\ y(t) = Cz(t), & \forall t \in [0, \tau]. \end{cases} \tag{1}$$

where z is the state and y the output function (where the dot symbol is used to denote the time derivative). Such systems are often used as models of vibrating systems.

The inverse problem considered here is to reconstruct the initial state $z_0 = z(0)$ of system (1) knowing *the observation* $y(t)$ on the time interval $[0, \tau]$ (see Figure 1). Such inverse problems arise in many applications, like thermoacoustic tomography (see Kuchment and Kunyansky [2]) or data assimilation (see Puel [3], Auroux and Blum [4]). To solve this inverse problem, we assume here that it is well-posed. More precisely we assume that (A, C) is exactly observable in time $\tau > 0$, i.e. that there exists $k_\tau > 0$ such that

$$\int_0^\tau \|y(t)\|^2 dt \geq k_\tau^2 \|z_0\|^2, \forall z_0 \in \mathcal{D}(A).$$

Our method is based on the construction of forward and backward observers associated with (1), which we define now. Following Liu [5, Theorem 2.3.], we know that $A^+ = A - C^*C$ (respectively $A^- = -A - C^*C$) generate an exponentially stable C_0-semigroup $\mathbb{T}^+$ (respectively $\mathbb{T}^-$) on X. Then, we introduce the following initial and final Cauchy problems, called respectively *forward* and *backward observers* of (1)

$$\begin{cases} \dot{z}^+(t) = A^+ z^+(t) + C^* y(t), & \forall t \in [0, \tau], \\ z^+(0) = 0, \end{cases} \tag{2}$$

$$\begin{cases} \dot{z}^-(t) = -A^- z^-(t) - C^* y(t), & \forall t \in [0, \tau], \\ z^-(\tau) = z^+(\tau). \end{cases} \tag{3}$$

Note that the states z^+ and z^- of the forward and backward observers are completely determined by the knowledge of the output y. If we set $\mathbb{L}_\tau = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$, then by [1, Proposition 3.7], we have $\eta := \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$ and by [1, Proposition 3.3], the following remarkable relation holds true

$$z_0 = (I - \mathbb{L}_\tau)^{-1} z^-(0).$$

In particular, one can invert the operator $(I - \mathbb{L}_\tau)$ using a Neumann series and get the following expression for the initial state

$$z_0 = \sum_{n=0}^{\infty} \mathbb{L}_\tau^n z^-(0). \tag{4}$$

Thus, at least theoretically, the reconstruction of the initial state is given by the above formula. Note that the computation of the first term in the above sum requires to solve the two non-homogeneous systems (2) and (3), while the terms for $n \geq 1$ involve the resolution of the two homogeneous systems associated with (2) and (3) (i.e. for $y \equiv 0$). In practice, the reconstruction procedure requires the discretization of these two systems and the truncation of the infinite sum in (4) to keep only a finite number of back and forth iterations. For instance, if we consider a full discretization described by a mesh size h (typically a finite element approximation) and a time step Δt, one can compute

$$z_{0,h,\Delta t} = \sum_{n=0}^{N_{h,\Delta t}} \mathbb{L}_{h,\Delta t,K}^n \left(z_h^-\right)^0.$$

where

- $\mathbb{L}_{h,\Delta t,K} = \mathbb{T}^-_{h,\Delta t,K}\mathbb{T}^+_{h,\Delta t,K}$, where $\mathbb{T}^\pm_{h,\Delta t,K}$ are suitable space and time discretizations of $\mathbb{T}^\pm_\tau$,
- $\left(z_h^-\right)^0 \in X_h$ is an approximation of $z^-(0)$,
- $N_{h,\Delta t}$ is a suitable truncation parameter.

Our objective in this work is to propose a convergence analysis of $z_{0,h,\Delta t}$ towards z_0. A particular attention will be devoted to the optimal choice of the truncation parameter $N_{h,\Delta t}$ for given discretization parameters (mesh size h and time step Δt). Let us emphasize that our error estimates (see (13)) provide in particular an upper bound for the maximum admissible noise under which convergence of the algorithm is guaranteed. As usually for approximation theory of PDE's, some regularity assumptions are needed to obtain error estimates. Namely, our result allows us to reconstruct only initial data contained in some subspace of X (namely $\mathcal{D}\left(A^2\right)$). Moreover, our analysis only holds for locally distributed observation (leading to bounded observation operators). The case of boundary observation (leading to unbounded observation operators) is open.

Wave type systems

Let H be a Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. The corresponding norm of H is denoted by $\|\cdot\|$. Let $A_0 : \mathcal{D}(A_0) \to H$ be a strictly positive self-adjoint operator and $C_0 \in \mathcal{L}(H,Y)$ a bounded observation operator, where Y is an other Hilbert space. The norm in $\mathcal{D}(A_0^\alpha)$ will be denoted by $\|\cdot\|_\alpha$. Given $\tau > 0$, we deal with the general wave type system

$$\begin{cases} \ddot{w}(t) + A_0 w(t) = 0, \quad \forall t \geq 0, \\ y(t) = C_0 \dot{w}(t), \quad \forall t \in [0,\tau], \end{cases} \tag{5}$$

and we want to reconstruct the initial value $(w_0, w_1) = (w(0), \dot{w}(0))$ of (5) knowing $y(t)$ for $t \in [0,\tau]$. In order to use the general iterative algorithm described in the introduction, we first rewrite (5) as a first order system of the form (1). To achieve this, it suffices to introduce the following notation:

$$z(t) = \begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}, \qquad X = \mathcal{D}\left(A_0^{\frac{1}{2}}\right) \times H,$$

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \qquad \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}\left(A_0^{\frac{1}{2}}\right), \tag{6}$$

$$C \in \mathcal{L}(X,Y), \qquad C = \begin{bmatrix} 0 & C_0 \end{bmatrix}. \tag{7}$$

The space X is endowed with the norm

$$\|z\| = \sqrt{\|z_1\|_{\frac{1}{2}}^2 + \|z_2\|^2}, \quad \forall z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in X.$$

We assume that the pair (A, C) is exactly observable in time $\tau > 0$. Thus, according to Liu [5, Theorem 2.3.], $A^+ = A - C^*C$ (resp. $A^- = -A - C^*C$) is the generator of an exponentially stable C_0-semigroup $\mathbb{T}^+$ (resp. $\mathbb{T}^-$). We set $\mathbb{L}_\tau = \mathbb{T}^-_\tau \mathbb{T}^+_\tau$. We also assume that $(w_0, w_1) \in \mathcal{D}\left(A^2\right) = \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$. Thus by applying Theorem 4.1.6 of Tucsnak and Weiss [6], we have

$$w \in C\left([0,\tau], \mathcal{D}\left(A_0^{\frac{3}{2}}\right)\right) \cap C^1\left([0,\tau], \mathcal{D}(A_0)\right) \cap C^2\left([0,\tau], \mathcal{D}\left(A_0^{\frac{1}{2}}\right)\right).$$

The forward and backward observers (2) and (3) read then as follows (as second-order systems)

$$\begin{cases} \ddot{w}^+(t) + A_0 w^+(t) + C_0^* C_0 \dot{w}^+(t) \\ \qquad = C_0^* y(t), \quad \forall t \in [0,\tau], \\ w^+(0) = 0, \quad \dot{w}^+(0) = 0, \end{cases} \tag{8}$$

$$\begin{cases} \ddot{w}^-(t) + A_0 w^-(t) - C_0^* C_0 \dot{w}^-(t) \\ \qquad = -C_0^* y(t), \quad \forall t \in [0,\tau], \\ w^-(\tau) = w^+(\tau), \quad \dot{w}^-(\tau) = \dot{w}^+(\tau). \end{cases} \tag{9}$$

Clearly, the above two systems can be written in a common abstract initial value Cauchy problem (simply by using a time reversal for the second system)

$$\begin{cases} \ddot{p}(t) + A_0 p(t) + C_0^* C_0 \dot{p}(t) = f(t), \quad \forall t \in [0,\tau], \\ p(0) = p_0, \quad \dot{p}(0) = p_1 \end{cases} \tag{10}$$

where we have set

- for the forward observer (8) : $f(t) = C_0^* y(t) = C_0^* C_0 \dot{w}(t)$ and $(p_0, p_1) = (0, 0)$,
- for the backward observer (9) : $f(t) = -C_0^* y(\tau - t) = -C_0^* C_0 \dot{w}(\tau - t)$ and $(p_0, p_1) = (w^+(\tau), -\dot{w}^+(\tau)) \in \mathcal{D}(A^2)$.

Let us emphasize that with these notation, the semigroups $\mathbb{T}^\pm$ are given by the relations

$$\mathbb{T}_t^+ \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix}, \qquad \mathbb{T}_t^- \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} p(\tau - t) \\ -\dot{p}(\tau - t) \end{bmatrix}$$

where p solves (10) with $f = 0$.

Given $(p_0, p_1) \in \mathcal{D}(A^2)$, the variational formulation of (10) reads for all $t \in [0, \tau]$ and all $\varphi \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ as follows

$$\begin{cases} \langle \ddot{p}(t), \varphi \rangle + \langle p(t), \varphi \rangle_{\frac{1}{2}} + \langle C_0^* C_0 \dot{p}(t), \varphi \rangle = \langle f(t), \varphi \rangle, \\ p(0) = p_0, \quad \dot{p}(0) = p_1. \end{cases} \tag{11}$$

In order to approximate (11) in space and time, we use an implicit finite difference scheme in time combined with a Galerkin method in space. More precisely, consider a family $(H_h)_{h>0}$ of finite-dimensional subspaces of $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ endowed with the norm in H. We denote π_h the orthogonal projection from $\mathcal{D}\left(A_0^{\frac{1}{2}}\right)$ onto H_h. We assume that there exist $M > 0$, $\theta > 0$ and $h^* > 0$ such that we have for all $h \in (0, h^*)$

$$\|\pi_h \varphi - \varphi\| \leq M h^\theta \|\varphi\|_{\frac{1}{2}}, \qquad \forall \varphi \in \mathcal{D}\left(A_0^{\frac{1}{2}}\right).$$

We discretize the time interval $[0, \tau]$ using a time step $\Delta t > 0$. We obtain a discretization $t_k = k\Delta t$, where $0 \leq k \leq K$ and where we assumed, without loss of generality, that $\tau = K\Delta t$. Given a function of time f of class $\mathcal{C}^2$, we approximate its first and second derivative at time t_k by

$$f'(t_k) \simeq D_t f(t_k) := \frac{f(t_k) - f(t_{k-1})}{\Delta t}.$$

$$f''(t_k) \simeq D_{tt} f(t_k) := \frac{f(t_k) - 2f(t_{k-1}) + f(t_{k-2})}{\Delta t^2}.$$

We suppose that $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) \in H_h \times H_h$ and f_h^k, for $0 \leq k \leq K$, are given approximations of (p_0, p_1) and $f(t_k)$ in the space X and H respectively. We define the approximate solution $(p_h^k)_{0 \leq k \leq K}$ of (11) as the solution of the following problem: $p_h^k \in H_h$ such that for all $\varphi_h \in H_h$ and all $2 \leq k \leq K$

$$\begin{cases} \left\langle D_{tt} p_h^k, \varphi_h \right\rangle + \left\langle p_h^k, \varphi_h \right\rangle_{\frac{1}{2}} \\ \qquad + \left\langle C_0^* C_0 D_t p_h^k, \varphi_h \right\rangle = \left\langle f_h^k, \varphi_h \right\rangle, \\ p_h^0 = p_{0,h,\Delta t}, \quad p_h^1 = p_h^0 + \Delta t\, p_{1,h,\Delta t}. \end{cases} \tag{12}$$

Note that the above procedure leads to a natural approximation $\mathbb{T}_{h,\Delta t,k}^\pm$ of the continuous operators $\mathbb{T}_{t_k}^\pm$ by setting

$$\begin{cases} \mathbb{T}_{t_k}^+ \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \simeq \mathbb{T}_{h,\Delta t,k}^+ \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} := \begin{bmatrix} p_h^k \\ D_t p_h^k \end{bmatrix}, \\ \\ \mathbb{T}_{t_k}^- \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \simeq \mathbb{T}_{h,\Delta t,k}^- \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} := \begin{bmatrix} p_h^{K-k} \\ -D_t p_h^{K-k} \end{bmatrix} \end{cases}$$

where p_h^k solves (12) with $f_h^k = 0$ for all $0 \leq k \leq K$ and for $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) = (\pi_h p_0, \pi_h p_1)$. Obviously, this leads to a fully discretized approximation of the operator $\mathbb{L}_\tau = \mathbb{T}_\tau^- \mathbb{T}_\tau^+$ by setting

$$\mathbb{L}_{h,\Delta t,K} = \mathbb{T}_{h,\Delta t,K}^- \mathbb{T}_{h,\Delta t,K}^+.$$

Assume that for all $0 \leq k \leq K$, y_h^k is a given approximation of $y(t_k)$ in Y and let $\left(w_h^+\right)^k$ and $\left(w_h^-\right)^k$ be respectively the approximations of (8) and (9) obtained via (12) as follows:

- For all $0 \leq k \leq K$, $\left(w_h^+\right)^k = p_h^k$ where p_h^k solves (12) with $f_h^k = C_0^* y_h^k$ and $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) = (0, 0)$,
- For all $0 \leq k \leq K$, $\left(w_h^-\right)^k = p_h^{K-k}$ where p_h^k solves (12) with $f_h^k = -C_0^* y_h^{K-k}$ and $(p_{0,h,\Delta t}, p_{1,h,\Delta t}) = ((w_h^+)^K, -D_t (w_h^+)^K)$.

Main results

Theorem 1. *Let $A_0 : \mathcal{D}(A_0) \to H$ be a strictly positive self-adjoint operator and $C_0 \in \mathcal{L}(H, Y)$ such that $C_0^* C_0 \in \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{3}{2}}\right)\right) \cap \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{1}{2}}\right)\right)$. Define (A, C) by* (6) *and* (7)*. Assume that the pair (A, C) is exactly observable in time $\tau > 0$ and set $\eta := \|\mathbb{L}_\tau\|_{\mathcal{L}(X)} < 1$. Let $(w_0, w_1) \in \mathcal{D}\left(A_0^{\frac{3}{2}}\right) \times \mathcal{D}(A_0)$ be the initial value of* (5) *and let* $(w_{0,h,\Delta t}, w_{1,h,\Delta t})$ *be defined by*

$$\begin{bmatrix} w_{0,h,\Delta t} \\ w_{1,h,\Delta t} \end{bmatrix} = \sum_{n=0}^{N_h} \mathbb{L}_{h,\Delta t,K}^{n} \begin{bmatrix} (w_h^-)^0 \\ D_t(w_h^-)^1 \end{bmatrix},$$

where $D_t(w_h^-)^1 = \dfrac{(w_h^-)^1 - (w_h^-)^0}{\Delta t}$.

Then there exist $M > 0$*,* $h^* > 0$ *and* $\Delta t^* > 0$ *such that for all* $h \in (0, h^*)$ *and* $\Delta t \in (0, \Delta t^*)$

$$\|w_0 - w_{0,h,\Delta t}\|_{\frac{1}{2}} + \|w_1 - w_{1,h,\Delta t}\| \leq M \left[\left(\frac{\eta^{N_{h,\Delta t}+1}}{1-\eta} + \left(h^\theta + \Delta t\right)(1+\tau)\, N_{h,\Delta t}^2 \right) \times \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 \right) + N_{h,\Delta t} \Delta t \sum_{\ell=0}^{K} \left\| C_0^*(y(t_\ell) - y_h^\ell) \right\| \right].$$

A particular choice of $N_{h,\Delta t}$ leads to an explicit error estimate (with respect to h and Δt) as shown in the next Corollary

Corollary 2. *Under the assumptions of Theorem 1, we set*

$$N_{h,\Delta t} = \frac{\ln(h^\theta + \Delta t)}{\ln \eta}.$$

Then, there exist $M_\tau > 0$*,* $h^* > 0$ *and* $\Delta t^* > 0$ *such that for all* $h \in (0, h^*)$ *and* $\Delta t \in (0, \Delta t^*)$

$$\|w_0 - w_{0,h,\Delta t}\|_{\frac{1}{2}} + \|w_1 - w_{1,h,\Delta t}\| \leq M_\tau \left[(h^\theta + \Delta t) \ln^2(h^\theta + \Delta t) \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 \right) + \left| \ln(h^\theta + \Delta t) \right| \Delta t \sum_{\ell=0}^{K} \left\| C_0^* \left(y(t_\ell) - y_h^\ell \right) \right\| \right]. \quad (13)$$

Numerical experiments

We used the iterative algorithm to reconstruct the initial value of a 1-D wave equation on $[0, 1]$ with Dirichlet boundary condition (the string equation with fixed ends). The initial data to be reconstructed are chosen such that an explicit solution of the forward wave equation is available, in order to avoid the inverse crime. The available observation is supposed to be the velocity of the string on the interval $[0, \frac{1}{10}]$, so that the system is observable in time $T = 2$ (even less, see Bardos, Lebeau and Rauch [7] and Liu [5]). In our talk, we will provide numerical experiments showing the efficiency of our method. In particular, we will investigate the influence of the gain coefficient (a constant parameter multiplying the observation operator) and the robustness to noise.

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