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ASYMPTOTIC STABILITY OF THE MULTIDIMENSIONAL WAVE EQUATION COUPLED WITH CLASSES OF POSITIVE-REAL IMPEDANCE BOUNDARY CONDITIONS

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Abstract. This paper proves the asymptotic stability of the multidimensional wave equation posed on a bounded open Lipschitz set, coupled with various classes of positive-real impedance boundary conditions, chosen for their physical relevance: time-delayed, standard diffusive (which includes the Riemann-Liouville fractional integral) and extended diffusive (which includes the Caputo fractional derivative). The method of proof consists in formulating an abstract Cauchy problem on an extended state space using a dissipative realization of the impedance operator, be it finite or infinite-dimensional. The asymptotic stability of the corresponding strongly continuous semigroup is then obtained by verifying the sufficient spectral conditions derived by Arendt and Batty (Trans. Amer. Math. Soc., 306 (1988)) as well as Lyubich and Vũ (Studia Math., 88 (1988)).

1. **Introduction.** The broad focus of this paper is the asymptotic stability of the wave equation with so-called impedance boundary conditions (IBCs), also known as acoustic boundary conditions.

Herein, the impedance operator, related to the Neumann-to-Dirichlet map, is assumed to be continuous linear time-invariant, so that it reduces to a time-domain convolution. *Passive* convolution operators [7, § 3.5], the kernels of which have a positive-real Laplace transform, find applications in physics in the modeling of locally-reacting energy absorbing material, such as non perfect conductors in electromagnetism [68] and liners in acoustics [47]. As a result, IBCs are commonly used with Maxwell's equations [29], the linearized Euler equations [47], or the wave equation [58].

Two classes of convolution operators are well-known due to the ubiquity of the physical phenomena they model. Slowly decaying kernels, which yield so-called *long-memory* operators, arise from losses without propagation (due to e.g. viscosity or electrical/thermal resistance); they include fractional kernels. On the other hand, lossless propagation, encountered in acoustical cavity for instance, can be represented as a *time delay*. Both effects can be combined, so that time-delayed long-memory operators model a propagation with losses.

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Stabilization of the wave equation by a boundary damping, as opposed to an internal damping, has been investigated in a wealth of works, most of which employing the equivalent admittance formulation (5), see Remark 2 for the terminology. Unless otherwise specified, the works quoted below deal with the multidimensional wave equation.

Early studies established exponential stability with a proportional admittance [10, 33, 32]. A delay admittance is considered in [51], where exponential stability is proven under a sufficient delay-independent stability condition that can be interpreted as a passivity condition of the admittance operator. The proof of well-posedness relies on the formulation of an evolution problem using an infinite-dimensional realization of the delay through a transport equation (see [20, § VI.6] [13, § 2.4] and references therein) and stability is obtained using observability inequalities. The addition of a 2-dimensional realization to a delay admittance has been considered in [54], where both exponential and asymptotic stability results are shown under a passivity condition using the energy multiplier method. See also [65] for a monodimensional wave equation with a non-passive delay admittance, where it is shown that exponential stability can be achieved provided that the delay is a multiple of the domain back-and-forth traveling time.

A class of space-varying admittance with finite-dimensional realizations have received thorough scrutiny in [1] for the monodimensional case and [2] for the multidimensional case. In particular, asymptotic stability is shown using the Arendt-Batty-Lyubich-Vũ (ABLV) theorem in an extended state space.

Admittance kernels defined by a Borel measure on $(0, \infty)$ have been considered in [11], where exponential stability is shown under an integrability condition on the measure [11, Eq. (7)]. This result covers both distributed and discrete time delays, as well as a class of integrable kernels. Other classes of integrable kernels have been studied in [16, 53, 35]. Integrable kernels coupled with a 2-dimensional realization are considered in [35] using energy estimates. Kernels that are both completely monotone and integrable are considered in [16], which uses the ABLV theorem on an extended state space, and in [53] with an added time delay, which uses the energy method to prove exponential stability. The energy multiplier method is also used in [4] to prove exponential stability for a class of non-integrable singular kernels.

The works quoted so far do not cover fractional kernels, which are non-integrable, singular, and completely monotone. As shown in [44], asymptotic stability results with fractional kernels can be obtained with the ABLV theorem by using their realization; two works that follow this methodology are [45], which covers the monodimensional Webster-Lokshin equation with a rational IBC, and [24], which covers a monodimensional wave equation with a fractional admittance.

The objective of this paper is to prove the asymptotic stability of the multidimensional wave equation (2) coupled with a wide range of IBCs (3) chosen for their physical relevance. All the considered IBCs share a common property: the Laplace transform of their kernel is a positive-real function. A common method of proof, inspired by [45], is employed that consists in formulating an abstract Cauchy problem on an extended state space (8) using a realization of each impedance operator, be it finite or infinite-dimensional; asymptotic stability is then obtained with the ABLV theorem, although a less general alternative based on the invariance principle is also discussed. In spite of the apparent unity of the approach, we are not able to provide a single, unified proof: this leads us to formulate a conjecture at the end of this work, which we hope will motivate further works.

This paper is organized as follows. Section 2 introduces the model considered, recalls some known facts about positive-real functions, formulates the ABLV theorem as Corollary 8, and establishes a preliminary well-posedness result in the Laplace domain that is the cornerstone of the stability proofs. The remaining sections demonstrate the applicability of Corollary 8 to IBCs with infinite-dimensional realizations that arise in physical applications. Delay IBCs are covered in Section 3, standard diffusive IBCs (e.g. fractional integral) are covered in Section 4, while extended diffusive IBCs (e.g. fractional derivative) are covered in Section 5. The extension of the obtained asymptotic stability results to IBCs that contain a first-order derivative term is carried out in Section 6.

Notation. Vector-valued quantities are denoted in bold, e.g. $\boldsymbol{f}$. The canonical scalar product in $\mathbb{C}^d$, $d \in [\![1,\infty[\![$, is denoted by $(\boldsymbol{f},\boldsymbol{g})_{\mathbb{C}^d} \coloneqq \sum_{i=1}^{d} f_i\overline{g_i}$, where $\overline{g_i}$ is the complex conjugate. Throughout the paper, scalar products are antilinear with respect to the second argument. Gradient and divergence are denoted by

$$\nabla f \coloneqq [\partial_i f]_{i\in[\![1,d]\!]}\,,\ \operatorname{div} \boldsymbol{f} \coloneqq \sum_{i=1}^{d} \partial_i f_i,$$

where ∂_i is the weak derivative with respect to the i-th coordinate. The scalar product (resp. norm) on a Hilbert space H is denoted by $(\cdot,\cdot)_H$ (resp. $\|\cdot\|_H$). The only exception is the space of square integrable functions $(L^2(\Omega))^d$, with $\Omega \subset \mathbb{R}^d$ open set, for which the space is omitted, i.e.

$$(\boldsymbol{f},\boldsymbol{g}) \coloneqq \int_\Omega (\boldsymbol{f}(\boldsymbol{x}),\boldsymbol{g}(\boldsymbol{x}))_{\mathbb{C}^d}\,\mathrm{d}\boldsymbol{x},\ \|\boldsymbol{f}\| \coloneqq \sqrt{(\boldsymbol{f},\boldsymbol{f})}.$$

The scalar product on $(H^1(\Omega))^d$ is

$$(\boldsymbol{f},\boldsymbol{g})_{H^1(\Omega)} \coloneqq (\boldsymbol{f},\boldsymbol{g}) + (\nabla \boldsymbol{f},\nabla \boldsymbol{g}).$$

The topological dual of a Hilbert space H is denoted by $H^{'}$, and L^2 is used as a pivot space so that for instance

$$H^{\frac{1}{2}} \subset L^2 \simeq (L^2)^{'} \subset H^{-\frac{1}{2}},$$

which leads to the following repeatedly used identity, for $p \in L^2$ and $\psi \in H^{\frac{1}{2}}$,

$$\langle p,\psi\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} = \langle p,\psi\rangle_{(L^2)',L^2} = (p,\overline{\psi})_{L^2}, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket (linear in both arguments).

Remark 1. All the Hilbert spaces considered in this paper are over $\mathbb{C}$.

Other commonly used notations are $\mathbb{R}^* \coloneqq \mathbb{R}\backslash\{0\}$, $\Re(s)$ (resp. $\Im(s)$) for the real (resp. imaginary) part of $s \in \mathbb{C}$, $A^\intercal$ for the transpose of a matrix A, $R(A)$ (resp. $\ker(A)$) for the range (resp. kernel) of A, $\mathcal{C}(\Omega)$ for the space of continuous functions, $\mathcal{C}_0^\infty(\Omega)$ for the space of infinitely smooth and compactly supported functions, $\mathcal{D}^{'}(\Omega)$ for the space of distributions (dual of $\mathcal{C}_0^\infty(\Omega)$), $\mathcal{E}^{'}(\Omega)$ for the space of compactly supported distributions, $\mathcal{L}(H)$ for the space of continuous linear operators over H, $\overline{\Omega}$ for the closure of Ω, $Y_1 : \mathbb{R} \to \{0,1\}$ for the Heaviside function (1 over $(0,\infty)$, null elsewhere), and δ for the Dirac distribution.

2. **Model, strategy, and preliminary results.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. The Cauchy problem considered in this paper is the wave equation under one of its first-order form, namely

$$\partial_t \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} + \begin{pmatrix} \nabla p \\ \operatorname{div} \boldsymbol{u} \end{pmatrix} = \boldsymbol{0} \quad \text{on } \Omega, \tag{2}$$

where $\boldsymbol{u}(t, \boldsymbol{x}) \in \mathbb{C}^d$ and $p(t, \boldsymbol{x}) \in \mathbb{C}$. To (2) is associated the so-called *impedance boundary condition* (IBC), formally defined as a time-domain convolution between p and $\boldsymbol{u} \cdot \boldsymbol{n}$,

$$p = z \star \boldsymbol{u} \cdot \boldsymbol{n} \quad \text{a.e. on } \partial\Omega, \tag{3}$$

where $\boldsymbol{n}$ is the unit outward normal and z is the impedance kernel. In general, z is a causal distribution, i.e. $z \in \mathcal{D}'_+(\mathbb{R})$, so that the convolution is to be understood in the sense of distributions [59, Chap. III] [30, Chap. IV].

This paper proves the asymptotic stability of strong solutions of the evolution problem (2,3) with an impedance kernel z whose positive-real Laplace transform is given by

$$\hat{z}(s) = \left(z_0 + z_\tau e^{-\tau s}\right) + z_1 s + \hat{z}_{\text{diff},1}(s) + s\,\hat{z}_{\text{diff},2}(s) \quad (\Re(s) > 0), \tag{4}$$

where $\tau > 0$, $z_\tau \in \mathbb{R}$, $z_0 \geq |z_\tau|$, $z_1 > 0$, and $z_{\text{diff},1}$ as well as $z_{\text{diff},2}$ are both locally integrable completely monotone kernels. The motivation behind the definition of this kernel is physical as it models passive systems that arise in e.g. electromagnetics [21], viscoelasticity [17, 41], and acoustics [28, 37, 48].

By assumption, the right-hand side of (4) is a sum of positive-real kernels that each admit a dissipative realization. This property enables to prove asymptotic stability with (4) by treating each of the four positive-real kernel separately: this is carried out in Sections 3–6. This modularity property enables to keep concise notation by dealing with the difficulty of each term one by one; it is illustrated in Section 6. As already mentioned in the introduction, the similarity between the four proofs leads us to formulate a conjecture at the end of the paper.

The purpose of the remainder of this section is to present the strategy employed to establish asymptotic stability as well as to prove preliminary results. Section 2.1 justifies why, in order to obtain a well-posed problem in L^2, the Laplace transform of the impedance kernel must be a *positive-real* function. Section 2.2 details the strategy used to establish asymptotic stability. Section 2.3 proves a consequence of the Rellich identity that is then used in Section 2.4 to obtain a well-posedness result on the Laplace-transformed wave equation, which will be used repeatedly.

Remark 2 (Terminology)**.** The boundary condition (3) can equivalently be written as

$$\boldsymbol{u} \cdot \boldsymbol{n} = y \star p \quad \text{a.e. on } \partial\Omega, \tag{5}$$

where y is known as the *admittance* kernel ($y \star z = \delta$, where δ is the Dirac distribution). This terminology can be justified, for example, by the acoustical application: an acoustic impedance is homogeneous to a pressure divided by a velocity. The asymptotic stability results obtained in this paper still hold by replacing the impedance by the admittance (in particular, the statement "$z \neq 0$" becomes "$y \neq 0$"). The third way of formulating (3), not considered in this paper, is the so-called *scattering* formulation [7, p. 89] [38, § 2.8]

$$p - \boldsymbol{u} \cdot \boldsymbol{n} = \beta \star (p + \boldsymbol{u} \cdot \boldsymbol{n}) \quad \text{a.e. on } \partial\Omega,$$

where β is known as the *reflection coefficient*. A Dirichlet boundary condition is recovered for $z = 0$ ($\beta = -\delta$) while a Neumann boundary condition is recovered for $y = 0$ ($\beta = +\delta$), so that the proportional IBC, obtained for $z = z_0\delta$ ($\beta = \frac{z_0-1}{z_0+1}\,\delta$), $z_0 \geq 0$, can be seen as an intermediate between the two.

Remark 3. The use of a convolution in (3) can be justified with the following classical result [59, § III.3] [7, Thm. 1.18]: if $\mathcal{Z}$ is a linear time-invariant and continuous mapping from $\mathcal{E}'(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$, then $\mathcal{Z}(u) = \mathcal{Z}(\delta) \star u$ for all $u \in \mathcal{E}'(\mathbb{R})$.

2.1. **Why positive-real kernels?** Assume that $(\boldsymbol{u}, p)$ is a strong solution, i.e. that it belongs to $\mathcal{C}([0,T];(H^1(\Omega))^{d+1})$. The elementary a priori estimate

$$\|(\boldsymbol{u},p)(T)\|^2 = \|(\boldsymbol{u},p)(0)\|^2 - 2\,\Re\left[\int_0^T (p(\tau), \boldsymbol{u}(\tau)\cdot\boldsymbol{n})_{L^2(\partial\Omega)}\,\mathrm{d}\tau\right] \tag{6}$$

suggests that to obtain a contraction semigroup, the impedance kernel must satisfy a passivity condition, well-known in system theory. This justifies why we restrict ourselves to impedance kernels that are *admissible* in the sense of the next definition, adapted from [7, Def. 3.3].

Definition 4 (Admissible impedance kernel)**.** A distribution $z \in \mathcal{D}'(\mathbb{R})$ is said to be an *admissible impedance kernel* if the operator $u \mapsto z \star u$ that maps $\mathcal{E}'(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$ enjoys the following properties: (i) causality, i.e. $z \in \mathcal{D}'_+(\mathbb{R})$; (ii) reality, i.e. real-valued inputs are mapped to real-valued outputs; (iii) passivity, i.e.

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}),\ \forall T > 0,\ \Re\left[\int_{-\infty}^T (z \star u(\tau), u(\tau))_{\mathbb{C}}\,\mathrm{d}\tau\right] \geq 0. \tag{7}$$

An important feature of admissible impedance kernels z is that their Laplace transforms $\hat{z}$ are *positive-real* functions, see Definition 5 and Proposition 6. Herein, the Laplace transform $\hat{z}$ is an analytic function on an open *right* half-plane, i.e.

$$\hat{z}(s) \coloneqq \int_0^\infty z(t)e^{-st}\,\mathrm{d}t \quad \left(s \in \mathbb{C}_c^+\right),$$

for some $c \geq 0$ with

$$\mathbb{C}_c^+ \coloneqq \{s \in \mathbb{C} \,|\, \Re(s) > c\}.$$

See [59, Chap. 6] and [7, Chap. 2] for the definition when $z \in \mathcal{D}'_+(\mathbb{R})$.

Definition 5 (Positive-real function)**.** A function $f : \mathbb{C}_0^+ \to \mathbb{C}$ is *positive-real* if f is analytic in $\mathbb{C}_0^+$, $f(s) \in \mathbb{R}$ for $s \in (0,\infty)$, and $\Re[f(s)] \geq 0$ for $s \in \mathbb{C}_0^+$.

Proposition 6. *A causal distribution $z \in \mathcal{D}'_+(\mathbb{R})$ is an admissible impedance kernel if and only if $\hat{z}$ is a positive-real function.*

Proof. See [38, § 2.11] for the case where the kernel $z \in L^1(\mathbb{R})$ is a function and [7, § 3.5] for the general case where $z \in \mathcal{D}'_+(\mathbb{R})$ is a causal distribution. (Note that, if z is an admissible impedance kernel, then z is also tempered.) □

Remark 7. The growth at infinity of positive-real functions is at most polynomial. More specifically, from the integral representation of positive-real functions [7, Eq. (3.21)], it follows that for $\Re(s) \geq c > 0$, $|\hat{z}(s)| \leq C(c)P(|s|)$ where P is a second degree polynomial.

2.2. **Abstract framework for asymptotic stability.** Let the causal distribution $z \in \mathcal{D}'_+(\mathbb{R})$ be an admissible impedance kernel. In order to prove the asymptotic stability of (2,3), we will use the following strategy in Sections 3–6. We first rely on the knowledge of a realization of the impedance operator $u \mapsto z \star u$ to formulate an abstract Cauchy problem on a Hilbert space H,

$$\dot{X}(t) = \mathcal{A}X,\ X(0) = X_0 \in H, \tag{8}$$

where the extended state X accounts for the memory of the IBC. The scalar product $(\cdot,\cdot)_H$ is defined using a Lyapunov functional associated with the realization. Since, by design, the problem has the energy estimate $\|X(t)\|_H \leq \|X_0\|_H$, it is natural to use the Lumer-Phillips theorem to show that the unbounded operator

$$\mathcal{A} :\ \mathcal{D}(\mathcal{A}) \subset H \to H \tag{9}$$

generates a strongly continuous semigroup of contractions on H, denoted by $\mathcal{T}(t)$. For initial data in $\mathcal{D}(\mathcal{A})$, the function

$$t \mapsto \mathcal{T}(t)X_0 \tag{10}$$

provides the unique strong solution in $\mathcal{C}([0,\infty);\mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0,\infty);H)$ of the evolution problem (8) [52, Thm. 1.3]. For (less regular) initial data in H, the solution is milder, namely $\mathcal{C}([0,\infty);H)$.

To prove the asymptotic stability of this solution, we rely upon the following result, where we denote by $\sigma(\mathcal{A})$ (resp. $\sigma_p(\mathcal{A})$) the spectrum (resp. point spectrum) of $\mathcal{A}$ [67, § VIII.1].

Corollary 8. *Let H be a complex Hilbert space and $\mathcal{A}$ be defined as (9). If*

(i) *$\mathcal{A}$ is dissipative, i.e. $\Re(\mathcal{A}X, X)_H \leq 0$ for every $X \in \mathcal{D}(\mathcal{A})$,*
(ii) *$\mathcal{A}$ is injective,*
(iii) *$s\mathcal{I} - \mathcal{A}$ is bijective for $s \in (0,\infty) \cup i\mathbb{R}^*$,*

then $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions $\mathcal{T}(t) \in \mathcal{L}(H)$ that is asymptotically stable, i.e.

$$\forall X_0 \in H,\ \|\mathcal{T}(t)X_0\|_H \underset{t\to\infty}{\longrightarrow} 0. \tag{11}$$

Proof. The Lumer-Phillips theorem, recalled in Theorem 51, shows that $\mathcal{A}$ generates a strongly continuous semigroup of contractions $\mathcal{T}(t) \in \mathcal{L}(H)$. In particular $\mathcal{A}$ is closed, from the Hille-Yosida theorem [52, Thm. 3.1], so that the resolvent operator $(s\mathcal{I} - \mathcal{A})^{-1}$ is closed whenever it is defined. A direct application of the closed graph theorem [67, § II.6] then yields

$$\{s \in \mathbb{C} \mid s\mathcal{I} - \mathcal{A} \text{ is bijective}\} \subset \rho(\mathcal{A}),$$

where $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$ [67, § VIII.1]. Hence $i\mathbb{R}^* \subset \rho(\mathcal{A})$ and Theorem 52 applies since $0 \notin \sigma_p(\mathcal{A})$. □

Remark 9. Condition (iii) of Corollary 8 could be loosened by only requiring that $s\mathcal{I} - \mathcal{A}$ be surjective for $s \in (0,\infty)$ and bijective for $s \in i\mathbb{R}^*$. However, in the proofs presented in this paper we always prove bijectivity for $s \in (0,\infty) \cup i\mathbb{R}^*$.

2.3. **A consequence of the Rellich identity.** Using the Rellich identity, we prove below that the Dirichlet and Neumann Laplacians do not have an eigenfunction in common.

Proposition 10. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. If $p \in H_0^1(\Omega)$ satisfies*

$$\forall \psi \in H^1(\Omega),\ (\nabla p, \nabla \psi) = \lambda(p, \psi) \tag{12}$$

for some $\lambda \in \mathbb{C}$, then $p = 0$ a.e. in Ω.

Proof. Let $p \in H_0^1(\Omega)$ be such that (12) holds for some $\lambda \in \mathbb{C}$. The proof is divided in two steps.

(a) Let us first assume that $\partial\Omega$ is $\mathcal{C}^\infty$. In particular,

$$\forall \psi \in H_0^1(\Omega),\ (\nabla p, \nabla \psi) = \lambda(p, \psi),$$

so that p is either null a.e. in Ω or an eigenfunction of the Dirichlet Laplacian. In the latter case, since the boundary $\partial\Omega$ is of class $\mathcal{C}^\infty$, we have the regularity result $p \in \mathcal{C}^\infty(\overline{\Omega})$ [22, Thm. 8.13]. An integration by parts then shows that, for $\psi \in H^1(\Omega)$,

$$(\partial_n p, \psi)_{L^2(\partial\Omega)} = (\Delta p + \lambda p, \psi) = 0,$$

so that $\partial_n p = 0$ in $\partial\Omega$. However since p is $\mathcal{C}^2(\overline{\Omega})$ and $\partial\Omega$ is smooth we have [56]

$$\|p\|^2 = \frac{\int_{\partial\Omega} (\partial_n p)^2 \partial_n(|\boldsymbol{x}|^2)\, \mathrm{d}\boldsymbol{x}}{4\lambda}, \tag{13}$$

which shows that $p = 0$ in Ω. (The spectrum of the Dirichlet Laplacian does not include 0 [22, § 8.12].)

(b) Let us now assume that $\partial\Omega$ is not $\mathcal{C}^\infty$. The strategy, suggested to us by Prof. Patrick Ciarlet, is to get back to (a) by extending p by zero. Let $\mathcal{B}$ be an open ball such that $\overline{\Omega} \subset \mathcal{B}$. We denote $\tilde{p}$ the extension of p by zero, i.e. $\tilde{p} = p$ on Ω with $\tilde{p}$ null on $\mathcal{B}\backslash\Omega$. From Proposition 49, we have $\tilde{p} \in H_0^1(\mathcal{B})$. Using the definition of $\tilde{p}$, we can write

$$\forall \psi \in H^1(\mathcal{B}),\ (\nabla \tilde{p}, \nabla \psi)_{L^2(\mathcal{B})} = \left(\nabla p, \nabla \left[\psi_{|\Omega}\right]\right)_{L^2(\Omega)} = \lambda \left(p, \psi_{|\Omega}\right)_{L^2(\Omega)} = \lambda(\tilde{p}, \psi)_{L^2(\mathcal{B})},$$

so that applying (a) to $\tilde{p} \in H_0^1(\mathcal{B})$ gives $\tilde{p} = 0$ a.e. in $\mathcal{B}$. □

2.4. **A well-posedness result in the Laplace domain.** The following result will be used repeatedly. We define

$$\overline{\mathbb{C}_0^+} \coloneqq \{s \in \mathbb{C} \,|\, \Re(s) \geq 0\}.$$

Theorem 11. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz boundary. Let $z : \overline{\mathbb{C}_0^+}\backslash\{0\} \to \mathbb{C}_0^+$ be such that $z(s) \in \mathbb{R}$ for $s \in (0, \infty)$. For every $s \in \overline{\mathbb{C}_0^+}\backslash\{0\}$ and $l \in H^{-1}(\Omega)$ there exists a unique $p \in H^1(\Omega)$ such that*

$$\forall \psi \in H^1(\Omega),\ (\nabla p, \nabla \psi) + s^2(p, \psi) + \frac{s}{z(s)}(p, \psi)_{L^2(\partial\Omega)} = \overline{l(\psi)}. \tag{14}$$

Moreover, there is $C(s) > 0$, independent of p, such that

$$\|p\|_{H^1(\Omega)} \leq C(s)\, \|l\|_{H^{-1}(\Omega)}.$$

Remark 12. Note that $s \mapsto z(s)$ need not be continuous, so that Theorem 11 can be used pointwise, i.e. for only some $s \in \overline{\mathbb{C}_0^+}\backslash\{0\}$.

Remark 13 (Intuition)**.** Although the need for Theorem 11 will appear in the proofs of the next sections, let us give a *formal* motivation for the formulation (14). Assume that $(\boldsymbol{u}, p)$ is a smooth solution of (2,3). Then p solves the wave equation

$$\partial_t^2 p - \Delta p = 0 \quad \text{on } \Omega,$$

with the impedance boundary condition

$$\partial_t p = z \star \partial_t \boldsymbol{u} \cdot \boldsymbol{n} = z \star (-\nabla p) \cdot \boldsymbol{n} = -z \star \partial_n p \quad \text{on } \partial\Omega,$$

where $\partial_n p$ denotes the normal derivative of p and the causal kernel z is, say, tempered and locally integrable. An integration by parts with $\psi \in H^1(\Omega)$ reads

$$(\nabla p, \nabla \psi) + (\partial_t^2 p, \psi) - (\partial_n p, \psi)_{L^2(\partial\Omega)} = 0.$$

The formulation (14) then follows from the application of the Laplace transform in time, which gives $\widehat{z \star \partial_n p}(s) = \hat{z}(s)\partial_n \hat{p}(s)$ and $\widehat{\partial_t p}(s) = s\hat{p}(s)$ assuming that $p(t = 0) = 0$ on $\partial\Omega$.

Proof for $s \in (0, \infty)$. If $s \in (0, \infty)$ this is an immediate consequence of the Lax-Milgram lemma [34, Thm. 6.6]. Define the following bilinear form over $H^1(\Omega) \times H^1(\Omega)$:

$$\overline{a(p, \psi)} \coloneqq (\nabla p, \nabla \psi) + s^2 (p, \psi) + \frac{s}{z(s)} (p, \psi)_{L^2(\partial\Omega)}.$$

Its boundedness follows from the continuity of the trace $H^1(\Omega) \to L^2(\partial\Omega)$ (see Section A.2). The fact that $z(s) > 0$ gives

$$a(\psi, \psi) \geq \min(1, s^2) \|\psi\|^2_{H^1(\Omega)},$$

which establishes the coercivity of a. □

Proof. Let $s \in \overline{\mathbb{C}_0^+} \backslash \{0\}$. The Lax-Milgram lemma does not apply since the sign of $\Re(\bar{s} z(s))$ is indefinite in general, but the Fredholm alternative is applicable. Using the Riesz-Fréchet representation theorem [34, Thm. 6.4], (14) can be rewritten uniquely as

$$(\mathcal{I} - \mathcal{K}(s))p = L \quad \text{in } H^1(\Omega), \tag{15}$$

where $L \in H^1(\Omega)$ satisfies $\overline{l(\psi)} = (L, \psi)_{H^1(\Omega)}$ and the operator $\mathcal{K}(s) \in \mathcal{L}(H^1(\Omega))$ is given by

$$(\mathcal{K}(s)p, \psi)_{H^1(\Omega)} \coloneqq (1 - s^2)(p, \psi) - \frac{s}{z(s)} (p, \psi)_{L^2(\partial\Omega)}.$$

The interest of (15) lies in the fact that $\mathcal{K}(s)$ turns out to be a compact operator, see Lemma 14. The Fredholm alternative states that $\mathcal{I} - \mathcal{K}(s)$ is injective if and only if it is surjective [8, Thm. 6.6]. Using Lemma 15 and the open mapping theorem [67, § II.5], we conclude that $\mathcal{I} - \mathcal{K}(s)$ is a bijection with continuous inverse, which yields the claimed well-posedness result. □

Lemma 14. *Let $s \in \overline{\mathbb{C}_0^+} \backslash \{0\}$. The operator $\mathcal{K}(s) \in \mathcal{L}(H^1(\Omega))$ is compact.*

Proof. Let $p, \psi \in H^1(\Omega)$. The Cauchy–Schwarz inequality and the continuity of the trace $H^1(\Omega) \to L^2(\partial\Omega)$ yield the existence of a constant $C > 0$ such that

$$\left|(\mathcal{K}(s)p, \psi)_{H^1(\Omega)}\right| \leq \left(\left|1 - s^2\right| \|p\| + C \left| \frac{s}{z(s)} \right| \|p\|_{L^2(\partial\Omega)} \right) \|\psi\|_{H^1(\Omega)},$$

from which we deduce

$$\|\mathcal{K}(s)p\|_{H^1(\Omega)} \leq \left|1 - s^2\right| \|p\| + C \left| \frac{s}{z(s)} \right| \|p\|_{L^2(\partial\Omega)}.$$

Let $\epsilon \in (0, \frac{1}{2})$. The continuous embedding $H^{\frac{1}{2}+\epsilon}(\Omega) \subset L^2(\Omega)$ and the continuity of the trace $H^{\frac{1}{2}+\epsilon}(\Omega) \to L^2(\partial\Omega)$, see Section A.2, yield

$$\|\mathcal{K}(s)p\|_{H^1(\Omega)} \leq \left(\left|1 - s^2\right| + C' \left| \frac{s}{z(s)} \right| \right) \|p\|_{H^{\epsilon+\frac{1}{2}}(\Omega)}.$$

The compactness of the embedding $H^1(\Omega) \subset H^{\frac{1}{2}+\epsilon}(\Omega)$, see Section A.2, enables to conclude. $\square$

Lemma 15. *Let $s \in \overline{\mathbb{C}_0^+}\backslash\{0\}$. The operator $\mathcal{I} - \mathcal{K}(s)$ is injective.*

Proof. Assume that $\mathcal{I} - \mathcal{K}(s)$ is not injective. Then there exists $p \in H^1(\Omega)\backslash\{0\}$ such that $\mathcal{K}(s)p = p$, i.e. for any $\psi \in H^1(\Omega)$,

$$(\nabla p, \nabla \psi) + s^2(p, \psi) + \frac{s}{z(s)}(p, \psi)_{L^2(\partial\Omega)} = 0. \tag{16}$$

In particular, for $\psi = p$,

$$z(s)\|\nabla p\|^2 + s^2 z(s)\|p\|^2 + s\|p\|^2_{L^2(\partial\Omega)} = 0. \tag{17}$$

To derive a contradiction, we distinguish between $s \in \mathbb{C}_0^+$ and $s \in i\mathbb{R}^*$.
($s \in \mathbb{C}_0^+$) This is a direct consequence of Lemma 16.
($s \in i\mathbb{R}^*$) Let $s = i\omega$ with $\omega \in \mathbb{R}^*$. Then (17) reads

$$\begin{cases} \Re(z(i\omega))\left(\|\nabla p\|^2 - \omega^2\|p\|^2\right) = 0 \\ \Im(z(i\omega))\left(\|\nabla p\|^2 - \omega^2\|p\|^2\right) + \omega\|p\|^2_{L^2(\partial\Omega)} = 0, \end{cases}$$

so that $p \in H_0^1(\Omega)$. Going back to the first identity (16), we therefore have

$$\forall \psi \in H^1(\Omega),\ (\nabla p, \nabla \psi) = \omega^2(p, \psi).$$

The contradiction then follows from Proposition 10. $\square$

Lemma 16. *Let $(a_0, a_1, a_2) \in [0, \infty)^3$ and $z \in \mathbb{C}_0^+$. The polynomial $s \mapsto z a_2\, s^2 + a_1\, s + z a_0$ has no roots in $\mathbb{C}_0^+$.*

Proof. The only case that needs investigating is $a_i > 0$ for $i \in [\![0, 2]\!]$. Let us denote by $\sqrt{\cdot}$ the branch of the square root that has a nonnegative real part, with a cut on $(-\infty, 0]$ (i.e. $\sqrt{\cdot}$ is analytic over $\mathbb{C}\backslash(-\infty, 0]$). The roots are given by

$$s_\pm := \frac{a_1 \overline{z}}{2a_2|z|^2}\left(-1 \pm \sqrt{1 - \gamma z^2}\right)$$

with $\gamma := 4\frac{a_0 a_2}{a_1^2} > 0$ so that

$$\Re(s_\pm) = \frac{a_1}{2a_2|z|^2} f_\pm(z) \quad \text{with} \quad f_\pm(z) := \Re\left[\overline{z}\left(-1 \pm \sqrt{1 - \gamma z^2}\right)\right].$$

The function $f_\pm$ is continuous on $\mathbb{C}_0^+\backslash[\gamma^{-1/2}, \infty)$ (but not analytic) and vanishes only on $i\mathbb{R}$ (if $f_\pm(z) = 0$, then there is $\omega \in \mathbb{R}$ such that $2\omega z = i\left(\omega^2 - \gamma|z|^4\right)$). The claim therefore follows from

$$f_\pm\left(\frac{1}{\sqrt{2\gamma}}\right) = \frac{-\sqrt{2} \pm 1}{2\sqrt{\gamma}} < 0.$$

$\square$

In view of Theorem 11, in the remainder of this paper, we make the following assumption on the set Ω.

Assumption 17. *The set $\Omega \subset \mathbb{R}^d$, $d \in [\![1, \infty[\![$, is a bounded open set with a Lipschitz boundary.*

3. **Delay impedance.** This section, as well as Sections 4 and 5, deals with IBCs that have an *infinite*-dimensional realization, which arise naturally in physical modeling [48]. Let us first consider the time-delayed impedance

$$\hat{z}(s) \coloneqq z_0 + z_\tau e^{-\tau s}, \tag{18}$$

where $z_0, z_\tau, \tau \in \mathbb{R}$, so that the corresponding IBC (3) reads

$$p(t) = z_0 \boldsymbol{u}(t) \cdot \boldsymbol{n} + z_\tau \boldsymbol{u}(t-\tau) \cdot \boldsymbol{n} \quad \text{a.e. on } \partial\Omega,\ t > 0. \tag{19}$$

The function (18) is positive-real if and only if

$$z_0 \geq |z_\tau|,\ \tau \geq 0, \tag{20}$$

which is assumed in the following. From now on, in addition to (20), we further assume

$$\hat{z}(0) \neq 0,\ \tau \neq 0.$$

This section is organized as follows: a realization of $\hat{z}$ is recalled in Section 3.1 and the stability of the coupled system is shown in Section 3.2.

Remark 18. In [51], exponential (resp. asymptotic) stability is shown under the condition $z_0 > z_\tau > 0$ (resp. $z_0 \geq z_\tau > 0$) and $\tau > 0$.

Remark 19. The case of a (memoryless) proportional impedance $\hat{z}(s) \coloneqq z_0$ with $z_0 > 0$ is elementary (it is known that exponential stability is achieved [10, 33, 32]) and can be covered by the strategy detailed in Section 2.2 without using an extended state space [49, § 4.2.2].

3.1. **Time-delay realization.** Following a well-known device, time-delays can be realized using a transport equation on a bounded interval [13, § 2.4] [20, § VI.6]. Let u be a causal input. The linear time-invariant operator $u \mapsto z \star u$ can be realized as

$$z \star u(t) = z_0 u(t) + z_\tau \chi(t, -\tau) \quad (t > 0),$$

where the state $\chi \in H^1(-\tau, 0)$ with $t \geq 0$ follows the transport equation

$$\begin{cases} \partial_t \chi(t,\theta) = \partial_\theta \chi(t,\theta), & (\theta \in (-\tau, 0),\ t > 0)\,, & \text{(a)} \\ \chi(0,\theta) = 0, & (\theta \in (-\tau, 0))\,, & \text{(b)} \\ \chi(t,0) = u(t), & (t > 0)\,. & \text{(c)} \end{cases} \tag{21}$$

For $\chi \in \mathcal{C}^1([0,T]; H^1(-\tau, 0))$ solution of (21a), we have the following energy balance

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi(t,\cdot)\|^2_{L^2(-\tau,0)} &= \Re(\partial_\theta \chi(t,\cdot), \chi(t,\cdot))_{L^2(-\tau,0)} \\ &= \frac{1}{2}\left[|\chi(t,0)|^2 - |\chi(t,-\tau)|^2\right], \end{aligned}$$

which we shall use in the proof of Lemma 23.

Remark 20 (Multiple delays)**.** Note that a finite number of time-delays $\tau_i > 0$ can be accounted for by setting $\tau \coloneqq \max_i \tau_i$ and writing

$$z \star u(t) = z_0 u(t) + \sum_i z_{\tau_i} \chi(t, -\tau_i).$$

The corresponding impedance $\hat{z}(s) = z_0 + \sum_i z_{\tau_i} e^{-\tau_i s}$ is positive-real if $z_0 \geq \sum_i |z_{\tau_i}|$. No substantial change to the proofs of Section 3.2 is required to handle this case. In [51], asymptotic stability is proven under the condition $z_0 \geq \sum_i z_i > 0$ and $z_i > 0$.

3.2. **Asymptotic stability.** Let

$$H_{\text{div}}(\Omega) \coloneqq \left\{ \boldsymbol{u} \in L^2(\Omega)^d \mid \operatorname{div} \boldsymbol{u} \in L^2(\Omega) \right\}.$$

The state space is defined as

$$\begin{aligned} H &\coloneqq \nabla H^1(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega; L^2(-\tau,0)), \\ ((\boldsymbol{u},p,\chi),(\boldsymbol{f_u},f_p,f_\chi))_H &\coloneqq (\boldsymbol{u},\boldsymbol{f_u}) + (p,f_p) + k(\chi,f_\chi)_{L^2(\partial\Omega;L^2(-\tau,0))}, \end{aligned} \tag{22}$$

where $k \in \mathbb{R}$ is a constant to be tuned to achieve dissipativity, see Lemma 23. The evolution operator is defined as

$$\begin{aligned} &\mathcal{D}(\mathcal{A}) \ni X \coloneqq \begin{pmatrix} \boldsymbol{u} \\ p \\ \chi \end{pmatrix} \longmapsto \mathcal{A}X \coloneqq \begin{pmatrix} -\nabla p \\ -\operatorname{div} \boldsymbol{u} \\ \partial_\theta \chi \end{pmatrix}, \\ &\mathcal{D}(\mathcal{A}) \coloneqq \left\{ (\boldsymbol{u},p,\chi) \in H \;\middle|\; \begin{array}{l} (\boldsymbol{u},p) \in H_{\text{div}}(\Omega) \times H^1(\Omega) \\ \chi \in L^2(\partial\Omega; H^1(-\tau,0)) \\ p = z_0 \boldsymbol{u}\cdot\boldsymbol{n} + z_\tau \chi(\cdot,-\tau) \text{ in } H^{-\frac{1}{2}}(\partial\Omega) \\ \chi(\cdot,0) = \boldsymbol{u}\cdot\boldsymbol{n} \text{ in } H^{-\frac{1}{2}}(\partial\Omega) \end{array} \right\}. \end{aligned} \tag{23}$$

In this formulation, the IBC (19) is the third equation in $\mathcal{D}(\mathcal{A})$. We apply Corollary 8, see the Lemmas 23, 24, and 25 below. Lemma 23 shows that the seemingly free parameter k must be restricted for $\|\cdot\|_H$ to be a Lyapunov functional, as formally highlighted in [46].

Remark 21 (Bochner's integral)**.** For the integrability of vector-valued functions, we follow the definitions and results presented in [67, § V.5]. Let $\mathcal{B}$ be a Banach space. We have [67, Thm. V.5.1]

$$L^2(\partial\Omega;\mathcal{B}) = \left\{ f : \partial\Omega \to \mathcal{B} \text{ strongly measurable} \;\middle|\; \|f\|_{\mathcal{B}} \in L^2(\partial\Omega) \right\}.$$

In Sections 4 and 5, we repeatedly use the following result: if $A \in \mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)$ and $u \in L^2(\partial\Omega;\mathcal{B}_1)$, then $Au \in L^2(\partial\Omega;\mathcal{B}_2)$.

Remark 22. Since $\nabla H^1(\Omega)$ is a closed subspace of $L^2(\Omega)^d$, H is a Hilbert space, see Section A.3 for some background. In view of the orthogonal decomposition (73), working with $\nabla H^1(\Omega)$ instead of $L^2(\Omega)^d$ enables to get an injective evolution operator $\mathcal{A}$. The exclusion of the solenoidal fields $\boldsymbol{u}$ that belong to $H_{\text{div}\,0,0}(\Omega)$ from the domain of $\mathcal{A}$ can be physically justified by the fact that these fields are non-propagating and are not affected by the IBC.

Lemma 23. *The operator $\mathcal{A}$ given by (23) is dissipative if and only if*

$$k \in \left[z_0 - \sqrt{z_0^2 - z_\tau^2},\, z_0 + \sqrt{z_0^2 - z_\tau^2} \right].$$

Proof. Let $X \in \mathcal{D}(\mathcal{A})$. In particular, $\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega)$ since $\chi(\cdot,0) \in L^2(\partial\Omega)$. Using Green's formula (72)

$$\begin{aligned} \Re(\mathcal{A}X,X)_H &= -\Re\left[\langle \boldsymbol{u}\cdot\boldsymbol{n}, \overline{p} \rangle_{H^{-\frac{1}{2}}(\partial\Omega),H^{\frac{1}{2}}(\partial\Omega)} \right] + k\,\Re\left(\partial_\theta\chi,\chi\right)_{L^2(\partial\Omega;L^2(-\tau,0))} \\ &= -\Re\left[(\boldsymbol{u}\cdot\boldsymbol{n},p)_{L^2(\partial\Omega)} \right] + \frac{k}{2}\|\chi(\cdot,0)\|^2_{L^2(\partial\Omega)} - \frac{k}{2}\|\chi(\cdot,-\tau)\|^2_{L^2(\partial\Omega)}, \\ &= \left(\frac{k}{2} - z_0 \right) \|\chi(\cdot,0)\|^2_{L^2(\partial\Omega)} - \frac{k}{2}\|\chi(\cdot,-\tau)\|^2_{L^2(\partial\Omega)} \\ &\quad - z_\tau \Re\left[(\chi(\cdot,0),\chi(\cdot,-\tau))_{L^2(\partial\Omega)} \right], \end{aligned}$$

from which we deduce that $\mathcal{A}$ is dissipative if and only if the matrix

$$\begin{bmatrix} z_0 - \frac{k}{2} & \frac{z_\tau}{2} \\ \frac{z_\tau}{2} & \frac{k}{2} \end{bmatrix}$$

is positive semidefinite, i.e. if and only if its determinant and trace are nonnegative:

$$(2z_0 - k)k \geq z_\tau^2 \quad \text{and} \quad z_0 \geq 0.$$

The conclusion follows the expressions of the roots of $k \mapsto -k^2 + 2z_0 k - z_\tau^2$. $\square$

Lemma 24. *The operator $\mathcal{A}$ given by (23) is injective.*

Proof. Assume $X \in \mathcal{D}(\mathcal{A})$ satisfies $\mathcal{A}X = 0$, i.e. $\nabla p = \mathbf{0}$, $\operatorname{div} \boldsymbol{u} = 0$, and

$$\partial_\theta \chi(\boldsymbol{x}, \theta) = 0 \quad \text{a.e. in } \partial\Omega \times (-\tau, 0). \tag{24}$$

Hence $\chi(\boldsymbol{x}, \cdot)$ is constant with

$$\chi(\cdot, 0) = \chi(\cdot, -\tau) = \boldsymbol{u} \cdot \boldsymbol{n} \quad \text{a.e. in } \partial\Omega. \tag{25}$$

Green's formula (72) yields

$$\langle \boldsymbol{u} \cdot \boldsymbol{n}, \overline{p} \rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = 0,$$

and by combining with the IBC (i.e. the third equation in $\mathcal{D}(\mathcal{A})$) and (25)

$$\hat{z}(0) \|\boldsymbol{u} \cdot \boldsymbol{n}\|^2_{L^2(\partial\Omega)} = 0,$$

where we have used that $\boldsymbol{u} \cdot \boldsymbol{n} \in L^2(\partial\Omega)$ since $\chi(\cdot, 0) \in L^2(\partial\Omega)$. Since $\hat{z}(0) \neq 0$ we deduce that $\boldsymbol{u} \in H_{\operatorname{div} 0,0}(\Omega)$, hence $\boldsymbol{u} = \mathbf{0}$ from (73) and $\chi = 0$. The IBC gives $p = 0$ a.e. on $\partial\Omega$, hence $p = 0$ a.e. on Ω. $\square$

Lemma 25. *Let $\mathcal{A}$ be given by (23). Then, $s\mathcal{I} - \mathcal{A}$ is bijective for $s \in (0, \infty) \cup i\mathbb{R}^*$.*

Proof. Let $F \in H$ and $s \in (0, \infty) \cup i\mathbb{R}^*$. We seek a *unique* $X \in \mathcal{D}(\mathcal{A})$ such that $(s\mathcal{I} - \mathcal{A})X = F$, i.e.

$$\begin{cases} s\boldsymbol{u} + \nabla p = \boldsymbol{f_u} & \text{(a)} \\ sp + \operatorname{div} \boldsymbol{u} = f_p & \text{(b)} \\ s\chi - \partial_\theta \chi = f_\chi. & \text{(c)} \end{cases} \tag{26}$$

The proof, as well as the similar ones found in the next sections, proceeds in three steps.

(a) As a preliminary step, let us *assume* that (26) holds with $X \in \mathcal{D}(\mathcal{A})$. Equation (26c) can be uniquely solved as

$$\chi(\cdot, \theta) = e^{s\theta} \boldsymbol{u} \cdot \boldsymbol{n} + R(s, \partial_\theta) f_\chi(\cdot, \theta), \tag{27}$$

where we denote

$$R(s, \partial_\theta) f_\chi(\boldsymbol{x}, \theta) \coloneqq \left[Y_1 e^{s\cdot} \star f_\chi(\boldsymbol{x}, \cdot) \right](\theta) = \int_0^\theta e^{s(\theta - \tilde{\theta})} f_\chi(\boldsymbol{x}, \tilde{\theta}) \, \mathrm{d}\tilde{\theta}.$$

We emphasize that, in the remaining of the proof, $R(s, \partial_\theta)$ is merely a convenient notation: the operator "∂_θ" cannot be defined independently from $\mathcal{A}$ (see Remark 26 for a detailed explanation).

The IBC (i.e. the third equation in $\mathcal{D}(\mathcal{A})$) can then be written as

$$p = \hat{z}(s) \boldsymbol{u} \cdot \boldsymbol{n} + z_\tau R(s, \partial_\theta) f_\chi(\cdot, -\tau) \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega), \tag{28}$$

and this identity actually takes place in $L^2(\partial\Omega)$ since

$$\boldsymbol{x} \mapsto R(s, \partial_\theta) f_\chi(\boldsymbol{x}, -\tau) \in L^2(\partial\Omega).$$

Let $\psi \in H^1(\Omega)$. Combining $(\boldsymbol{f_u}, \nabla\psi) + s(f_p, \psi)$ with (28) yields

$$
\begin{aligned}
(\nabla p, \nabla\psi) + s^2(p,\psi) + \frac{s}{\hat{z}(s)}(p,\psi)_{L^2(\partial\Omega)} =& (\boldsymbol{f_u}, \nabla\psi) + s(f_p,\psi) \\
&+ \frac{s z_\tau}{\hat{z}(s)}(R(s,\partial_\theta) f_\chi(\cdot,-\tau), \psi)_{L^2(\partial\Omega)}.
\end{aligned} \tag{29}
$$

In summary, $(s\mathcal{I} - \mathcal{A})X = F$ with $X \in \mathcal{D}(\mathcal{A})$ implies (29).

(b) We now construct a state $X \in \mathcal{D}(\mathcal{A})$ such that $(s\mathcal{I} - \mathcal{A})X = F$. To do so, we use the conclusion from the preliminary step (a).

Let $p \in H^1(\Omega)$ be the unique solution of (29) obtained with Theorem 11. It remains to find suitable $\boldsymbol{u}$ and χ so that $(\boldsymbol{u}, p, \chi) \in \mathcal{D}(\mathcal{A})$. Let us *define* $\boldsymbol{u} \in \nabla H^1(\Omega)$ by (26a). Taking $\psi \in \mathcal{C}_0^\infty(\Omega)$ in (29) shows that $\boldsymbol{u} \in H_{\text{div}}(\Omega)$ with (26b). Using the expressions of $\boldsymbol{u}$ and $\operatorname{div}\boldsymbol{u}$, and Green's formula (72), the weak formulation (29) can be rewritten as

$$
\langle \boldsymbol{u}\cdot\boldsymbol{n}, \overline{\psi}\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = \frac{1}{\hat{z}(s)}(p,\psi)_{L^2(\partial\Omega)} - \frac{z_\tau}{\hat{z}(s)}(R(s,\partial_\theta) f_\chi(\cdot,-\tau), \psi)_{L^2(\partial\Omega)}.
$$

which shows that p and $\boldsymbol{u}$ satisfy (28). Let us now *define* χ in $L^2(\partial\Omega; H^1(-\tau, 0))$ by (27). By rewriting (28) as

$$
p = (\hat{z}(s) - z_\tau e^{-s\tau})\boldsymbol{u}\cdot\boldsymbol{n} + z_\tau \left(e^{-s\tau}\boldsymbol{u}\cdot\boldsymbol{n} + R(s,\partial_\theta) f_\chi(\cdot,-\tau)\right) \quad \text{in } H^{-\frac{1}{2}}(\partial\Omega),
$$

we deduce thanks to (18) and (27) that the IBC holds, i.e. that $(\boldsymbol{u}, p, \chi) \in \mathcal{D}(\mathcal{A})$.

(c) We now show the uniqueness in $\mathcal{D}(\mathcal{A})$ of a solution of (26). The uniqueness of p in $H^1(\Omega)$ follows from Theorem 11. Although $\boldsymbol{u}$ is not unique in $H_{\text{div}}(\Omega)$, it is unique in $H_{\text{div}}(\Omega) \cap \nabla H^1(\Omega)$ following (73). The uniqueness of χ follows from the fact that (26c) is uniquely solvable in $\mathcal{D}(\mathcal{A})$. □

Remark 26. In the proof, $R(s, \partial_\theta)$ is only a notation since ∂_θ (hence also its resolvent operator) cannot be defined separately from $\mathcal{A}$. Indeed, the definition of ∂_θ would be

$$
\left|\begin{aligned}
&\partial_\theta : \mathcal{D}(\partial_\theta) \subset L^2(\partial\Omega; L^2(-\tau,0)) \to L^2(\partial\Omega; L^2(-\tau,0)) \\
&\qquad \chi \mapsto \partial_\theta \chi,
\end{aligned}\right.
$$

with domain

$$
\mathcal{D}(\partial_\theta) = \left\{\chi \in L^2(\partial\Omega; H^1(-\tau,0)) \mid \chi(\cdot,0) = \boldsymbol{u}\cdot\boldsymbol{n}\right\}
$$

that depends upon $\boldsymbol{u}$.

4. **Standard diffusive impedance.** This section focuses on the class of so-called *standard diffusive* kernels [50], defined as

$$
z(t) := \int_0^\infty e^{-\xi t} Y_1(t)\, \mathrm{d}\mu(\xi), \tag{30}
$$

where $t \in \mathbb{R}$ and μ is a positive Radon measure on $[0,\infty)$ that satisfies the following well-posedness condition

$$
\int_0^\infty \frac{\mathrm{d}\mu(\xi)}{1+\xi} < \infty, \tag{31}
$$

which guarantees that $z \in L^1_{\text{loc}}([0,\infty))$ with Laplace transform

$$
\hat{z}(s) = \int_0^\infty \frac{1}{s+\xi} \mathrm{d}\mu(\xi). \tag{32}
$$

The estimate

$$\forall s \in \overline{\mathbb{C}_0^+}\backslash\{0\}, \quad \frac{1}{|s+\xi|} \leq \sqrt{2}\max\left[1, \frac{1}{|s|}\right]\frac{1}{1+\xi}, \tag{33}$$

which is used below, shows that $\hat{z}$ is defined on $\overline{\mathbb{C}_0^+}\backslash\{0\}$.

This class of (positive-real) kernels is physically linked to non-propagating lossy phenomena and arise in electromagnetics [21], viscoelasticity [17, 41], and acoustics [28, 37, 48]. Formally, $\hat{z}$ admits the following realization

$$\begin{cases} \partial_t \varphi(t,\xi) = -\xi\varphi(t,\xi) + u(t), \ \varphi(0,\xi) = 0 \quad (\xi \in (0,\infty)), \\ z \star u(t) = \displaystyle\int_0^\infty \varphi(t,\xi)\,\mathrm{d}\mu(\xi). \end{cases} \tag{34}$$

The realization (34) can be given a meaning using the theory of well-posed linear systems [66, 62, 42, 64]. However, in order to prove asymptotic stability, we need a framework to give a meaning to the *coupled* system (2,3,34), which, it turns out, can be done without defining a well-posed linear system out of (34).

Similarly to the previous sections, this section is divided into two parts. Section 4.1 defines the realization of (34) and establishes some of its properties. These properties are then used in Section 4.2 to prove asymptotic stability of the coupled system.

Remark 27. The typical standard diffusive operator is the Riemann-Liouville fractional integral [57, § 2.3] [43]

$$\hat{z}(s) = \frac{1}{s^\alpha}, \ \mathrm{d}\mu(\xi) = \frac{\sin(\alpha\pi)}{\pi}\frac{1}{\xi^\alpha}\mathrm{d}\xi, \tag{35}$$

where $\alpha \in (0,1)$.

Remark 28. The expression (30) arises naturally when inverting multivalued Laplace transforms, see [19, Chap. 4] for applications in partial differential equations. However, a standard diffusive kernel can also be defined as follows: a causal kernel z is said to be *standard diffusive* if it belongs to $L^1_{\mathrm{loc}}([0,\infty))$ and is completely monotone on $(0,\infty)$. By Bernstein's representation theorem [25, Thm. 5.2.5], z is standard diffusive iff (30,31) hold. Additionally, a standard diffusive kernel z is integrable on $(0,\infty)$ iff

$$\mu(\{0\}) = 0 \quad \text{and} \quad \int_0^\infty \frac{1}{\xi}\mathrm{d}\mu(\xi) < \infty,$$

a property which will be referred to in Section 4.1. State spaces for the realization of classes of completely monotone kernels have been studied in [17, 60].

4.1. **Abstract realization.** To give a meaning to (34) suited for our purpose, we define, for any $s \in \mathbb{R}$, the following Hilbert space

$$V_s := \left\{\varphi : (0,\infty) \to \mathbb{C} \text{ measurable} \;\middle|\; \int_0^\infty |\varphi(\xi)|^2(1+\xi)^s\,\mathrm{d}\mu(\xi) < \infty\right\},$$

with scalar product

$$(\varphi,\psi)_{V_s} := \int_0^\infty (\varphi(\xi),\psi(\xi))_{\mathbb{C}}(1+\xi)^s\,\mathrm{d}\mu(\xi),$$

so that the triplet (V_{-1}, V_0, V_1) satisfies the continuous embeddings

$$V_1 \subset V_0 \subset V_{-1}. \tag{36}$$

The space V_0 will be the energy space of the realization, see (46). Note that the spaces V_{-1} and V_1 defined above are different from those encountered when defining a well-posed linear system out of (34), see [42]. When $\mathrm{d}\mu$ is given by (35), the spaces V_0 and V_1 reduce to the spaces "H_α" and "V_α" defined in [45, § 3.2].

On these spaces, we wish to define the unbounded state operator A, the control operator B, and the observation operator C so that

$$A : \mathcal{D}(A) \coloneqq V_1 \subset V_{-1} \to V_{-1},\ B \in \mathcal{L}(\mathbb{C}, V_{-1}),\ C \in \mathcal{L}(V_1, \mathbb{C}). \tag{37}$$

The state operator is defined as the following multiplication operator

$$A : \left| \begin{array}{l} \mathcal{D}(A) \coloneqq V_1 \subset V_{-1} \to V_{-1} \\ \varphi \mapsto (\xi \mapsto -\xi\varphi(\xi)). \end{array} \right. \tag{38}$$

The control operator is simply

$$Bu \coloneqq \xi \mapsto u, \tag{39}$$

and belongs to $\mathcal{L}(\mathbb{C}, V_{-1})$ thanks to the condition (31) since, for $u \in \mathbb{C}$,

$$\|Bu\|_{V_{-1}} = \left[\int_0^\infty \frac{1}{1+\xi}\,\mathrm{d}\mu(\xi)\right]^{1/2} |u|.$$

The observation operator is

$$C\varphi \coloneqq \int_0^\infty \varphi(\xi)\,\mathrm{d}\mu(\xi),$$

and $C \in \mathcal{L}(V_1, \mathbb{C})$ thanks to (31) as, for $\varphi \in V_1$,

$$|C\varphi| \leq \left[\int_0^\infty \frac{1}{1+\xi}\,\mathrm{d}\mu(\xi)\right]^{1/2} \|\varphi\|_{V_1}.$$

The next lemma gathers properties of the triplet (A, B, C) that are used in Section 4.2 to obtain asymptotic stability. Recall that if A is closed and $s \in \rho(A)$, then the resolvent operator $R(s, A)$ defined by (75) belongs to $\mathcal{L}(V_{-1}, V_1)$ [31, § III.6.1].

Lemma 29. *The operator A defined by (38) is injective, generates a strongly continuous semigroup of contractions on V_{-1}, and satisfies $\overline{\mathbb{C}_0^+}\backslash\{0\} \subset \rho(A)$.*

Proof. The proof is split into three steps, (a), (b), and (c). (a) The injectivity of A follows directly from its definition. (b) Let us show that $(0,\infty) \cup i\mathbb{R}^* \subset \rho(A)$. Let $f_\varphi \in V_{-1}$, $s \in (0,\infty) \cup i\mathbb{R}^*$, and define

$$\varphi(\xi) \coloneqq \frac{1}{s+\xi} f_\varphi(\xi) \quad \text{a.e. on } (0,\infty). \tag{40}$$

Using the estimate (33), we have

$$\|\varphi\|_{V_1} \leq \sqrt{2} \max\left[1, \frac{1}{|s|}\right] \|f_\varphi\|_{V_{-1}},$$

so that φ belongs to V_1 and $(s\mathcal{I} - A)\varphi = f_\varphi$ is well-posed. (c) For any $\varphi \in V_1$, we have $\Re\left[(A\varphi, \varphi)_{V_{-1}}\right] \leq -\|\varphi\|^2_{V_0}$, so A is dissipative. By the Lumer-Phillips theorem, A generates a strongly continuous semigroup of contractions on V_{-1}, so that $\mathbb{C}_0^+ \subset \rho(A)$ [52, Cor. 3.6]. □

Lemma 30. *The triplet of operators (A, B, C) defined above satisfies (37) as well as the following properties.*

(i) *(Stability) A is closed and injective with $\overline{\mathbb{C}_0^+}\backslash\{0\} \subset \rho(A)$.*

(ii) *(Regularity)*
   (a) $A \in \mathcal{L}(V_1, V_{-1})$.
   (b) For any $s \in \overline{\mathbb{C}_0^+}\backslash\{0\}$,
$$AR(s,A)_{|V_0} \in \mathcal{L}(V_0, V_0), \tag{41}$$
   where the vertical line denotes the restriction.

(iii) *(Reality) For any* $s \in (0, \infty)$,
$$CR(s,A)B_{|\mathbb{R}} \in \mathbb{R}, \tag{42}$$

(iv) *(Passivity) For any* $(\varphi, u) \in \mathcal{D}(A\&B)$,
$$\Re\left[(A\varphi + Bu, \varphi)_{V_0} - (u, C\varphi)_{\mathbb{C}}\right] \leq 0, \tag{43}$$
where we define
$$\mathcal{D}(A\&B) \coloneqq \{(\varphi, u) \in V_1 \times \mathbb{C} \mid A\varphi + Bu \in V_0\}.$$

Proof. Let A, B, and C be defined as above. Each of the properties is proven below.

(i) This condition is satisfied from Lemma 29.

(iia) Let $\varphi \in V_1$. We have
$$\begin{aligned}\|A\varphi\|_{V_{-1}}^2 &= \int_0^\infty |\varphi(\xi)|^2 \frac{\xi^2}{1+\xi}\,\mathrm{d}\mu(\xi)\\ &\leq \int_0^\infty |\varphi(\xi)|^2 (1+\xi)\,\mathrm{d}\mu(\xi) = \|\varphi\|_{V_1}^2,\end{aligned}$$
using the inequality $\xi^2 \leq (1+\xi)^2$.

(iib) Let $f_\varphi \in V_0$ and $s \in \overline{\mathbb{C}_0^+}\backslash\{0\}$,
$$\|AR(s,A)f_\varphi\|_{V_0} = \left[\int_0^\infty \left|\frac{\xi}{s+\xi} f_\varphi\right|^2 \mathrm{d}\mu(\xi)\right]^{1/2} \leq \|f_\varphi\|_{V_0},$$
where we have used $\left|\frac{\xi}{s+\xi}\right| \leq \frac{\xi}{\Re(s)+\xi} \leq 1$.

(iii) Let $s \in (0, \infty)$ and $u \in \mathbb{R}$. The reality condition is fulfilled since
$$CR(s,A)Bu = u\int_0^\infty \frac{\mathrm{d}\mu(\xi)}{s+\xi}.$$

(iv) Let $(\varphi, u) \in \mathcal{D}(A\&B)$. We have
$$\Re\left[(A\varphi + Bu, \varphi)_{V_0} - (u, C\varphi)_{\mathbb{C}}\right] = -\Re\left[\int_0^\infty \xi|\varphi(\xi)|^2\,\mathrm{d}\mu(\xi)\right] \leq 0, \tag{44}$$
so that the passivity condition is satisfied.

□

Remark 31. The space $\mathcal{D}(A\&B)$ is nonempty. Indeed, it contains at least the following one dimensional subspace
$$\{(\varphi, u) \in V_1 \times \mathbb{C} \mid \varphi = R(s,A)Bu\}$$
for any $s \in \rho(A)$ (which is nonempty from Lemma 30(i)); this follows from
$$\begin{aligned}A\varphi + Bu =& AR(s,A)Bu + Bu\\ =& sR(s,A)Bu \in V_1.\end{aligned}$$
It also contains $\{(R(s,A)\varphi, 0) \mid \varphi \in V_0\}$.

For any $s \in \rho(A)$, we define

$$z \coloneqq s \mapsto CR(s, A)B, \tag{45}$$

which is analytic, from the analyticity of $R(\cdot, A)$ [31, Thm. III.6.7]. Additionally, we have $z(s) \in \mathbb{R}$ for $s \in (0, \infty)$ from (42), and $\Re(z(s)) \geq 0$ from the passivity condition (43) with $\varphi \coloneqq R(s, A)Bu \in \mathcal{D}(A\&B)$:

$$\Re(s)\|R(s, A)Bu\|_{V_0}^2 \leq \Re\left[(u, z(s)u)_{\mathbb{C}}\right].$$

Since $\mathbb{C}_0^+ \subset \rho(A)$, the function z defined by (45) is positive-real.

4.2. **Asymptotic stability.** Let (A, B, C) be defined as in Section 4.1. We further assume that A, B, and C are non-null operators. The coupling between the wave equation (2) and the infinite-dimensional realization (A, B, C) can be formulated as the abstract Cauchy problem (8) using the following definitions. The extended state space is

$$\begin{aligned} H &\coloneqq \nabla H^1(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega; V_0), \\ ((\boldsymbol{u}, p, \varphi), (\boldsymbol{f_u}, f_p, f_\varphi))_H &\coloneqq (\boldsymbol{u}, \boldsymbol{f_u}) + (p, f_p) + (\varphi, f_\varphi)_{L^2(\partial\Omega; V_0)}, \end{aligned} \tag{46}$$

and the evolution operator $\mathcal{A}$ is

$$\begin{gathered} \mathcal{D}(\mathcal{A}) \ni X \coloneqq \begin{pmatrix} \boldsymbol{u} \\ p \\ \varphi \end{pmatrix} \longmapsto \mathcal{A}X \coloneqq \begin{pmatrix} -\nabla p \\ -\operatorname{div} \boldsymbol{u} \\ A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n} \end{pmatrix}, \\ \mathcal{D}(\mathcal{A}) \coloneqq \left\{ (\boldsymbol{u}, p, \varphi) \in H \;\middle|\; \begin{aligned} &(\boldsymbol{u}, p, \varphi) \in H_{\mathrm{div}}(\Omega) \times H^1(\Omega) \times L^2(\partial\Omega; V_1) \\ &(A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n}) \in L^2(\partial\Omega; V_0) \\ &p = C\varphi \text{ in } H^{\frac{1}{2}}(\partial\Omega) \end{aligned} \right\}, \end{gathered} \tag{47}$$

where the IBC (3,34) is the third equation in $\mathcal{D}(\mathcal{A})$.

Remark 32. In the definition of $\mathcal{A}$, there is an abuse of notation. Indeed, we still denote by A the following operator

$$\left| \begin{aligned} &L^2(\partial\Omega; V_1) \to L^2(\partial\Omega; V_{-1}) \\ &\varphi \mapsto (\boldsymbol{x} \mapsto A\varphi(\boldsymbol{x}, \cdot)), \end{aligned} \right.$$

which is well-defined from Lemma 30(iia) and Remark 21. A similar abuse of notation is employed for B and C.

Asymptotic stability is proven by applying Corollary 8 through Lemmas 34, 35, and 36 below. In order to clarify the proofs presented in Lemmas 34 and 35, we first prove a regularity property on $\boldsymbol{u}$ that follows from the definition of $\mathcal{D}(\mathcal{A})$.

Lemma 33 (Boundary regularity)**.** *If $X = (\boldsymbol{u}, p, \varphi) \in \mathcal{D}(\mathcal{A})$, then $\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega)$.*

Proof. Let $X \in \mathcal{D}(\mathcal{A})$. By definition of $\mathcal{D}(\mathcal{A})$, we have $\varphi \in L^2(\partial\Omega; V_1)$ so that $A\varphi \in L^2(\partial\Omega; V_{-1})$ from Lemma 30(iia) and Remark 21. From

$$B\boldsymbol{u}\cdot\boldsymbol{n} = \underbrace{A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n}}_{\in L^2(\partial\Omega; V_0)} - \overbrace{A\varphi}^{\in L^2(\partial\Omega; V_{-1})},$$

we deduce that $B\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega; V_{-1})$. The conclusion then follows from the definition of B and the condition (31). $\square$

Lemma 34. *The operator $\mathcal{A}$ given by (47) is dissipative.*

Proof. Let $X \in \mathcal{D}(\mathcal{A})$. In particular, $\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega)$ from Lemma 33. Green's formula (72) and the inequality (43) yield

$$\begin{aligned}\Re(\mathcal{A}X, X)_H &= \Re\left[(A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n}, \varphi)_{L^2(\partial\Omega;V_0)} - \langle \boldsymbol{u}\cdot\boldsymbol{n}, \overline{p}\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}\right] \\ &= \Re\left[(A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n}, \varphi)_{L^2(\partial\Omega;V_0)} - (\boldsymbol{u}\cdot\boldsymbol{n}, C\varphi)_{L^2(\partial\Omega)}\right] \leq 0,\end{aligned}$$

where we have used that $\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega)$. □

Lemma 35. *The operator $\mathcal{A}$ given by (47) is injective.*

Proof. Assume $X \in \mathcal{D}(\mathcal{A})$ satisfies $\mathcal{A}X = 0$. In particular $\nabla p = \boldsymbol{0}$ and $\operatorname{div}\boldsymbol{u} = 0$, so that Green's formula (72) yields

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}, \overline{p}\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = 0,$$

and by combining with the IBC (i.e. the third equation in $\mathcal{D}(\mathcal{A})$)

$$(\boldsymbol{u}\cdot\boldsymbol{n}, C\varphi)_{L^2(\partial\Omega)} = 0, \tag{48}$$

where we have used that $\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega)$ from Lemma 33. The third equation that comes from $\mathcal{A}X = 0$ is

$$A\varphi(\boldsymbol{x},\cdot) + B\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) = 0 \quad \text{in } V_0 \text{ for a.e. } \boldsymbol{x}\in\partial\Omega. \tag{49}$$

We now prove that $X = 0$, the key step being solving (49). Since A is injective, (49) has at most one solution $\varphi \in L^2(\partial\Omega; V_1)$. Let us distinguish the possible cases.

- If $0 \in \rho(A)$, then $\varphi = R(0, A)B\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega; V_1)$ is the unique solution. Inserting in (48) and using (45) yields
$$(\boldsymbol{u}\cdot\boldsymbol{n}, z(0)\boldsymbol{u}\cdot\boldsymbol{n})_{L^2(\partial\Omega)} = 0,$$
from which we deduce that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ since $z(0)$ is non-null.
- If $0 \in \sigma_r(A)\cup\sigma_c(A)$, then either $\overline{R(A)} \neq V_{-1}$ (definition of the residual spectrum) or $\overline{R(A)} = V_{-1}$ but $R(A) \neq V_{-1}$ (definition of the continuous spectrum combined with the closed graph theorem, since A is closed). $R(A)$ is equipped with the norm from V_{-1}. If $B\boldsymbol{u}\cdot\boldsymbol{n} \notin L^2(\partial\Omega; R(A))$, then the only solution is $\varphi = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. If $B\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega; R(A))$, then $\varphi = -A^{-1}B\boldsymbol{u}\cdot\boldsymbol{n}$ is the unique solution, where $A^{-1} : R(A) \to V_1$ is an unbounded closed bijection. Inserting in (48) yields
$$(\boldsymbol{u}\cdot\boldsymbol{n}, (-CA^{-1}B)\boldsymbol{u}\cdot\boldsymbol{n})_{L^2(\partial\Omega)} = 0.$$
Since $(-CA^{-1}B) \in \mathbb{C}$ is non-null, we deduce that $\boldsymbol{u}\cdot\boldsymbol{n} = 0$.

In summary, $\boldsymbol{u} \in H_{\operatorname{div}0,0}(\Omega)$, $\varphi = 0$ in $L^2(\partial\Omega; V_1)$, and $p = 0$ in $L^2(\partial\Omega)$. The nullity of p follows from $\nabla p = 0$. The nullity of $\boldsymbol{u}$ follows from $H_{\operatorname{div}0,0}(\Omega) \cap \nabla H^1(\Omega) = \{0\}$, see (73).

□

Lemma 36. *Let $\mathcal{A}$ be given by (47). Then, $s\mathcal{I} - \mathcal{A}$ is bijective for $s \in (0,\infty)\cup i\mathbb{R}^*$.*

Proof. Let $F \in H$ and $s \in (0,\infty)\cup i\mathbb{R}^*$. We seek a *unique* $X \in \mathcal{D}(\mathcal{A})$ such that $(s\mathcal{I} - \mathcal{A})X = F$, i.e.

$$\begin{cases} s\boldsymbol{u} + \nabla p = \boldsymbol{f_u} & \text{(a)} \\ sp + \operatorname{div}\boldsymbol{u} = f_p & \text{(b)} \\ s\varphi - A\varphi - B\boldsymbol{u}\cdot\boldsymbol{n} = f_\varphi. & \text{(c)} \end{cases} \tag{50}$$

For later use, let us note that Equation (50c) and the IBC (i.e. the third equation in $\mathcal{D}(\mathcal{A})$) imply

$$\varphi = R(s,A)(B\boldsymbol{u}\cdot\boldsymbol{n} + f_\varphi) \quad \text{in } L^2(\partial\Omega; V_1) \tag{51}$$

$$p = z(s)\boldsymbol{u}\cdot\boldsymbol{n} + CR(s,A)f_\varphi \quad \text{in } L^2(\partial\Omega). \tag{52}$$

Let $\psi \in H^1(\Omega)$. Combining $(\boldsymbol{f_u}, \nabla\psi) + s(f_p, \psi)$ with (52) yields

$$\begin{aligned}(\nabla p, \nabla\psi) + s^2(p,\psi) + \frac{s}{z(s)}(p,\psi)_{L^2(\partial\Omega)} = {} & (\boldsymbol{f_u}, \nabla\psi) + s(f_p,\psi) \\ & + \frac{s}{z(s)}(CR(s,A)f_\varphi, \psi)_{L^2(\partial\Omega)}.\end{aligned} \tag{53}$$

Note that since $CR(s,A) \in \mathcal{L}(V_{-1}, \mathbb{C})$, we have

$$\boldsymbol{x} \mapsto CR(s,A)f_\varphi(\boldsymbol{x}) \in L^2(\partial\Omega),$$

so that (53) is meaningful. Moreover, we have $\Re(z(s)) \geq 0$, and $z(s) \in (0,\infty)$ for $s \in (0,\infty)$. Therefore, we can apply Theorem 11, pointwise, for $s \in (0,\infty) \cup i\mathbb{R}^*$.

Let us denote by p the unique solution of (53) in $H^1(\Omega)$, obtained from Theorem 11. It remains to find suitable $\boldsymbol{u}$ and φ.

Let us *define* $\boldsymbol{u} \in \nabla H^1(\Omega)$ by (50a). Taking $\psi \in \mathcal{C}_0^\infty(\Omega)$ in (53) shows that $\boldsymbol{u} \in H_{\text{div}}(\Omega)$ and (50b) holds. Using the expressions of $\boldsymbol{u}$ and $\operatorname{div}\boldsymbol{u}$, and Green's formula (72), the weak formulation (53) can be rewritten as

$$\langle \boldsymbol{u}\cdot\boldsymbol{n}, \overline{\psi}\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)} = z(s)^{-1}(p,\psi)_{L^2(\partial\Omega)} - z(s)^{-1}(CR(s,A)f_\varphi, \psi)_{L^2(\partial\Omega)},$$

which shows that p and $\boldsymbol{u}$ satisfy (52).

Let us now *define* φ with (51), which belongs to $L^2(\partial\Omega; V_1)$. By rewriting (52) as

$$p = (z(s) - CR(s,A)B)\boldsymbol{u}\cdot\boldsymbol{n} + CR(s,A)(B\boldsymbol{u}\cdot\boldsymbol{n} + f_\varphi),$$

we obtain from (45) and (51) that the IBC holds.

To obtain $(\boldsymbol{u}, p, \varphi) \in \mathcal{D}(\mathcal{A})$ it remains to show that $A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n}$ belongs to $L^2(\partial\Omega; V_0)$. Using the definition (51) of φ, we have

$$\begin{aligned}A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n} &= AR(s,A)(B\boldsymbol{u}\cdot\boldsymbol{n} + f_\varphi) + B\boldsymbol{u}\cdot\boldsymbol{n} \\ &= (AR(s,A) + \mathcal{I})B\boldsymbol{u}\cdot\boldsymbol{n} + AR(s,A)f_\varphi \\ &= sR(s,A)B\boldsymbol{u}\cdot\boldsymbol{n} + AR(s,A)f_\varphi.\end{aligned}$$

Since $\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega)$ and $R(s,A)B \in \mathcal{L}(\mathbb{C}, V_1)$, we have

$$sR(s,A)B\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega; V_1).$$

The property (41) implies that

$$AR(s,A)f_\varphi \in L^2(\partial\Omega; V_0),$$

hence that $(\boldsymbol{u}, p, \varphi) \in \mathcal{D}(\mathcal{A})$.

The uniqueness of p follows from Theorem 11, that of $\boldsymbol{u}$ from (73), and that of φ from the bijectivity of $sI - A$. ∎

Remark 37. The time-delay case does not fit into the framework proposed in Section 4.1, see Remark 26. This justifies why delay and standard diffusive IBCs are covered separately.

5. **Extended diffusive impedance.** In this section, we focus on a variant of the standard diffusive kernel, namely the so-called *extended diffusive* kernel given by

$$\hat{z}(s) \coloneqq \int_0^\infty \frac{s}{s+\xi} \mathrm{d}\mu(\xi), \tag{54}$$

where μ is a Radon measure that satisfies the condition (31), already encountered in the standard case, and

$$\int_0^\infty \frac{1}{\xi} \mathrm{d}\mu(\xi) = \infty. \tag{55}$$

The additional condition (55) implies that $t \mapsto \int_0^\infty e^{-\xi t}\,\mathrm{d}\mu(\xi)$ is not integrable on $(0,\infty)$, see Remark 28.

From (34), we directly deduce that $\hat{z}$ *formally* admits the realization

$$\begin{cases} \partial_t \varphi(t,\xi) = -\xi\varphi(t,\xi) + u(t),\ \varphi(0,\xi) = 0 \quad (\xi \in (0,\infty))\,, \\ z \star u(t) = \displaystyle\int_0^\infty (-\xi\varphi(t,\xi) + u(t))\,\mathrm{d}\mu(\xi), \end{cases} \tag{56}$$

where u is a causal input. The separate treatment of the standard (32) and extended (54) cases is justified by the fact that physical models typically yield non-integrable kernels, i.e.

$$\int_0^\infty \mathrm{d}\mu(\xi) = +\infty, \tag{57}$$

which prevents from splitting the observation integral in (56): the observation and feedthrough operators must be combined into $C\&D$. This justifies why (56) is only formal. Although a functional setting for (56) has been obtained in [46, § B.3], we shall again follow the philosophy laid out in Section 4. Namely, Section 5.1 presents an abstract realization framework whose properties are given in Lemma 41, which slightly differs from the standard case, and Section 5.2 shows asymptotic stability of the coupled system (66).

Remark 38. Let $\alpha \in (0,1)$. The typical extended diffusive operator is the Riemman-Liouville fractional derivative [55, § 2.3] [43], obtained for $\hat{z}(s) = s^{1-\alpha}$ and $\mathrm{d}\mu$ given by (35), which satisfies the condition (55). For this measure $\mathrm{d}\mu$, choosing the initialization $\varphi(0,\xi) = {}^{u(0)}/_{\xi}$ in (56) yields the Caputo derivative [37].

5.1. **Abstract realization.** To give meaning to the realization (56) we follow a similar philosophy to the standard case, namely the definition of a triplet of Hilbert spaces (V_{-1}, V_0, V_1) that satisfies the continuous embeddings (36) as well as a suitable triplet of operators (A, B, C).

The Hilbert spaces V_{-1}, V_0, and V_1 are defined as

$$\begin{aligned} V_1 &\coloneqq \left\{ \varphi :\ (0,\infty) \to \mathbb{C} \text{ measurable} \;\middle|\; \int_0^\infty |\varphi(\xi)|^2 (1+\xi)\,\mathrm{d}\mu(\xi) < \infty \right\} \\ V_0 &\coloneqq \left\{ \varphi :\ (0,\infty) \to \mathbb{C} \text{ measurable} \;\middle|\; \int_0^\infty |\varphi(\xi)|^2 \xi\,\mathrm{d}\mu(\xi) < \infty \right\} \\ V_{-1} &\coloneqq \left\{ \varphi :\ (0,\infty) \to \mathbb{C} \text{ measurable} \;\middle|\; \int_0^\infty |\varphi(\xi)|^2 \frac{\xi}{1+\xi^2}\,\mathrm{d}\mu(\xi) < \infty \right\}, \end{aligned}$$

with scalar products

$$(\varphi,\psi)_{V_1} \coloneqq \int_0^\infty (\varphi(\xi),\psi(\xi))_{\mathbb{C}}(1+\xi)\,\mathrm{d}\mu(\xi)$$
$$(\varphi,\psi)_{V_0} \coloneqq \int_0^\infty (\varphi(\xi),\psi(\xi))_{\mathbb{C}}\,\xi\,\mathrm{d}\mu(\xi)$$
$$(\varphi,\psi)_{V_{-1}} \coloneqq \int_0^\infty (\varphi(\xi),\psi(\xi))_{\mathbb{C}}\frac{\xi}{1+\xi^2}\,\mathrm{d}\mu(\xi),$$

so that the continuous embeddings (36) are satisfied. Note the change of definition of the energy space V_0, which reflects the fact that the Lyapunov functional of (34) is different from that of (56): compare the energy balance (44) with (63). The change in the definition of V_{-1} is a consequence of this new definition of V_0. When $\mathrm{d}\mu$ is given by (35), the spaces V_0 and V_1 reduce to the spaces "$\tilde{H}_\alpha$" and "V_α" defined in [45, § 3.2].

The operators A, B, and C satisfy (contrast with (37))

$$A:\,\mathcal{D}(A)\coloneqq V_0\subset V_{-1}\to V_{-1},\ B\in\mathcal{L}(\mathbb{C},V_{-1}),\ C\in\mathcal{L}(V_1,\mathbb{C}). \tag{58}$$

The state operator A is still the multiplication operator (38), but with domain V_0 instead of V_1. Let us check that this definition makes sense. For any $\varphi\in V_0$, we have

$$\|A\varphi\|_{V_{-1}} = \left[\int_0^\infty |\varphi(\xi)|^2\frac{\xi^3}{1+\xi^2}\,\mathrm{d}\mu(\xi)\right]^{1/2} \leq \|\varphi\|_{V_0}. \tag{59}$$

The control operator B is defined as (39) and we have for any $u\in\mathbb{C}$

$$\|Bu\|_{V_{-1}} = \left[\int_0^\infty |u|^2\frac{\xi}{1+\xi^2}\,\mathrm{d}\mu(\xi)\right]^{1/2} \leq \tilde{C}\left[\int_0^\infty \frac{1}{1+\xi}\,\mathrm{d}\mu(\xi)\right]^{1/2}|u|,$$

where the constant $\tilde{C}>0$ is

$$\tilde{C}\coloneqq\left\|\frac{\xi(1+\xi)}{1+\xi^2}\right\|_{L^\infty(0,\infty)}.$$

The observation operator C is identical to the standard case. For use in Section 5.2, properties of (A,B,C) are gathered in Lemma 41 below.

Lemma 39. *The operator A generates a strongly continuous semigroup of contractions on V_{-1} and satisfies $\overline{\mathbb{C}_0^+}\backslash\{0\}\subset\rho(A)$.*

Proof. The proof is similar to that of Lemma 29. Let $s\in\overline{\mathbb{C}_0^+}\backslash\{0\}$ and $f_\varphi\in V_{-1}$. Let us define φ by (40). (a) We have

$$\begin{aligned}\|\varphi\|_{V_0} &= \left[\int_0^\infty \left|\frac{1}{s+\xi}f_\varphi\right|^2\xi\,\mathrm{d}\mu(\xi)\right]^{1/2}\\ &\leq \sqrt{2}\max\left[1,\frac{1}{|s|}\right]\left\|\frac{1+\xi^2}{(1+\xi)^2}\right\|_{L^\infty(0,\infty)}\|f_\varphi\|_{V_{-1}},\end{aligned}$$

so that φ solves $(s\mathcal{I}-A)\varphi=f_\varphi$ in V_0. Since $s\mathcal{I}-A$ is injective, we deduce that $s\in\rho(A)$. (b) Let $\varphi\in V_0$. We have

$$(A\varphi,\varphi)_{V_{-1}} = -\int_0^\infty |\varphi(\xi)|^2\frac{\xi^2}{1+\xi^2}\,\mathrm{d}\mu(\xi)\leq -\|\varphi\|^2_{V_0},$$

so that A is dissipative. The conclusion follows from the Lumer-Phillips theorem. $\square$

Lemma 40. *The operators A and B are injective. Moreover, if (55) holds, then $R(A) \cap R(B) = \{0\}$.*

Proof. The injectivity of A and B is immediate. Let $f_\varphi \in R(A) \cap R(B)$, so that there is $\varphi \in V_0$ and $u \in \mathbb{C}$ such that $A\varphi = Bu$, i.e. $-\xi\varphi(\xi) = u$ a.e. on $(0, \infty)$. The function φ belongs to V_0 if and only if

$$|u|^2 \int_0^\infty \frac{1}{\xi} \mathrm{d}\mu(\xi) < \infty.$$

So that, assuming (55), φ belongs to V_0 if and only if $u = 0$ a.e on $(0, \infty)$. $\square$

Lemma 41. *The triplet of operators (A, B, C) defined above satisfies (58) as well as the following properties.*

(i) *(Stability) A is closed with $\overline{\mathbb{C}_0^+}\backslash\{0\} \subset \rho(A)$ and satisfies*

$$\forall(\varphi, u) \in \mathcal{D}(C\&D),\ A\varphi = Bu \Rightarrow (\varphi, u) = (0, 0), \tag{60}$$

where we define

$$\mathcal{D}(C\&D) \coloneqq \{(\varphi, u) \in V_0 \times \mathbb{C} \mid A\varphi + Bu \in V_1\}.$$

(ii) *(Regularity)*
  (a) $A \in \mathcal{L}(V_0, V_{-1})$.
  (b) *For any $s \in \overline{\mathbb{C}_0^+}\backslash\{0\}$,*

$$AR(s, A)_{|V_0} \in \mathcal{L}(V_0, V_1),\ R(s, A)B \in \mathcal{L}(\mathbb{C}, V_1). \tag{61}$$

(iii) *(Reality) Identical to Lemma 30(iii).*

(iv) *(Passivity) For any $(\varphi, u) \in \mathcal{D}(C\&D)$,*

$$\Re\left[(A\varphi + Bu, \varphi)_{V_0} - (u, C(A\varphi + Bu))_{\mathbb{C}}\right] \leq 0. \tag{62}$$

Proof. Let (A, B, C) be as defined above. Each of the properties is proven below.

(i) Follows from Lemmas 39 and 40.

(iia) Follows from (59).

(iib) Let $s \in \overline{\mathbb{C}_0^+}\backslash\{0\}$, $f_\varphi \in V_0$, and $u \in \mathbb{C}$. We have

$$\begin{aligned}\|AR(s, A)f_\varphi\|_{V_1} &= \left[\int_0^\infty |f_\varphi(\xi)|^2 \frac{\xi^2(1+\xi)}{|s+\xi|^2}\,\mathrm{d}\mu(\xi)\right]^{1/2} \\ &\leq \sqrt{2}\max\left[1, \frac{1}{|s|}\right] \|f_\varphi\|_{V_0},\end{aligned}$$

and

$$\begin{aligned}\|R(s, A)Bu\|_{V_1} &= \left(\int_0^\infty \frac{1+\xi}{|s+\xi|^2}\,\mathrm{d}\mu(\xi)\right)^{1/2} |u| \\ &\leq \sqrt{2}\max\left[1, \frac{1}{|s|}\right] \left(\int_0^\infty \frac{1}{1+\xi}\,\mathrm{d}\mu(\xi)\right)^{1/2} |u|.\end{aligned}$$

(iii) Immediate.

(iv) Let $(\varphi, u) \in \mathcal{D}(C\&D)$. We have

$$
\begin{aligned}
\Re\Big[(A\varphi+Bu,\varphi)_{V_0} &- (u, C(A\varphi+Bu))_{\mathbb{C}}\Big] \\
&= \Re\left[\int_0^\infty (-\xi\varphi(\xi)+u,\varphi(\xi))_{\mathbb{C}}\,\xi\,\mathrm{d}\mu(\xi) - \left(u,\int_0^\infty(-\xi\varphi(\xi)+u)\,\mathrm{d}\mu(\xi)\right)_{\mathbb{C}}\right] \\
&= \Re\left[\int_0^\infty (-\xi\varphi(\xi)+u,\xi\varphi(\xi)-u)_{\mathbb{C}}\,\mathrm{d}\mu(\xi)\right] \\
&= -\Re\left[\int_0^\infty |-\xi\varphi(\xi)+u|^2\,\mathrm{d}\mu(\xi)\right] \leq 0. \qquad (63)
\end{aligned}
$$

□

The remarks made for the standard case hold identically (in particular, $\mathcal{D}(C\&D)$ is nonempty). For $s \in \rho(A)$ we define

$$
z(s) \coloneqq s\,CR(s,A)B. \tag{64}
$$

5.2. **Asymptotic stability.** Let (A, B, C) be the triplet of operators defined in Section 5.1, further assumed to be non-null. The abstract Cauchy problem (8) considered herein is the following. The state space is

$$
\begin{gathered}
H \coloneqq \nabla H^1(\Omega)\times L^2(\Omega)\times L^2(\partial\Omega;V_0), \\
((\boldsymbol{u},p,\varphi),(\boldsymbol{f_u},f_p,f_\varphi))_H \coloneqq (\boldsymbol{u},\boldsymbol{f_u})+(p,f_p)+(\varphi,f_\varphi)_{L^2(\partial\Omega;V_0)},
\end{gathered} \tag{65}
$$

and $\mathcal{A}$ is defined as

$$
\begin{gathered}
\mathcal{D}(\mathcal{A}) \ni X \coloneqq \begin{pmatrix} \boldsymbol{u} \\ p \\ \varphi \end{pmatrix} \longmapsto \mathcal{A}X \coloneqq \begin{pmatrix} -\nabla p \\ -\operatorname{div}\boldsymbol{u} \\ A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n} \end{pmatrix}, \\
\mathcal{D}(\mathcal{A}) \coloneqq \left\{ (\boldsymbol{u},p,\varphi)\in H \;\middle|\; \begin{array}{l} (\boldsymbol{u},p)\in H_{\mathrm{div}}(\Omega)\times H^1(\Omega) \\ (A\varphi+B\boldsymbol{u}\cdot\boldsymbol{n})\in L^2(\partial\Omega;V_1) \\ p = C(A\varphi+B\boldsymbol{u}\cdot\boldsymbol{n}) \text{ in } H^{\frac{1}{2}}(\partial\Omega) \end{array} \right\}.
\end{gathered} \tag{66}
$$

The technicality here is that the operator $(\varphi, u) \mapsto C(A\varphi + Bu)$ is defined over $\mathcal{D}(C\&D)$, but CB is not defined in general: this is the abstract counterpart of (57). An immediate consequence of the definition of $\mathcal{D}(\mathcal{A})$ is given in the following lemma.

Lemma 42 (Boundary regularity)**.** *If $X = (\boldsymbol{u}, p, \varphi) \in \mathcal{D}(\mathcal{A})$, then $\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega)$.*

Proof. Let $X \in \mathcal{D}(\mathcal{A})$. By definition of $\mathcal{D}(\mathcal{A})$, we have $\varphi \in L^2(\partial\Omega; V_0)$ so that $A\varphi \in L^2(\partial\Omega; V_{-1})$ from Lemma 41(iia) and Remark 21. The proof is then identical to that of Lemma 33. □

The application of Corollary 8 is summarized in the lemmas below, namely Lemmas 43, 44, and 45. Due to the similarities with the standard case, the proofs are more concise and focus on the differences.

Lemma 43. *The operator $\mathcal{A}$ defined by (66) is dissipative.*

Proof. Let $X \in \mathcal{D}(\mathcal{A})$. In particular, $\boldsymbol{u} \cdot \boldsymbol{n} \in L^2(\partial\Omega)$ from Lemma 42. Green's formula (72) and (62) yield

$$\Re(\mathcal{A}X, X)_H = \Re\Big[(A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n}, \varphi)_{L^2(\partial\Omega;V_0)} \\ - (\boldsymbol{u}\cdot\boldsymbol{n}, C(A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n}))_{L^2(\partial\Omega)}\Big] \leq 0,$$

using Lemma 41. □

The next proof is much simpler than in the standard case.

Lemma 44. *$\mathcal{A}$, given by (66), is injective.*

Proof. Assume $X \in \mathcal{D}(\mathcal{A})$ satisfies $\mathcal{A}X = 0$. In particular $\nabla p = \mathbf{0}$, $\operatorname{div} \boldsymbol{u} = 0$, and $A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n} = 0$ in $L^2(\partial\Omega; V_1)$. The IBC (i.e. the third equation in $\mathcal{D}(\mathcal{A})$) gives $p = 0$ in $L^2(\partial\Omega)$ hence $p = 0$ in $L^2(\Omega)$. From Lemma 42, $\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega)$ so we have at least $B\boldsymbol{u}\cdot\boldsymbol{n} \in L^2(\partial\Omega; V_{-1})$. Using (60), we deduce $\varphi = 0$ and $\boldsymbol{u}\cdot\boldsymbol{n} = 0$, hence $\boldsymbol{u} = 0$ from (73). □

Lemma 45. *$s\mathcal{I} - \mathcal{A}$, with $\mathcal{A}$ given by (66), is bijective for $s \in (0, \infty) \cup i\mathbb{R}^*$.*

Proof. Let $F \in H$, $s \in (0, \infty) \cup i\mathbb{R}^*$, and $\psi \in H^1(\Omega)$. We seek a *unique* $X \in \mathcal{D}(\mathcal{A})$ such that $(s\mathcal{I} - \mathcal{A})X = F$, i.e. (50), which implies

$$(\nabla p, \nabla\psi) + s^2(p, \psi) + \frac{s}{z(s)}(p, \psi)_{L^2(\partial\Omega)} = (\boldsymbol{f_u}, \nabla\psi) + s(f_p, \psi) \\ + \frac{s}{z(s)}(CAR(s, A)f_\varphi, \psi)_{L^2(\partial\Omega)}. \tag{67}$$

Note that, from (61), the right-hand side defines an anti-linear form on $H^1(\Omega)$. Let us denote by p the unique solution of (67) obtained from a pointwise application of Theorem 11 (we rely here on (42)). It remains to find suitable $\boldsymbol{u}$ and φ, in a manner identical to the standard diffusive case.

Taking $\psi \in \mathcal{C}_0^\infty(\Omega)$ in (67) shows that $\boldsymbol{u} \in H_{\mathrm{div}}(\Omega)$ with (50b). Using the expressions of $\boldsymbol{u} \in \nabla H^1(\Omega)$ and $\operatorname{div} \boldsymbol{u}$, and Green's formula (72), the weak formulation (67) shows that p and $\boldsymbol{u}$ satisfy, in $L^2(\partial\Omega)$,

$$p = z(s)\boldsymbol{u}\cdot\boldsymbol{n} + CAR(s, A)f_\varphi. \tag{68}$$

Let us now *define* φ as

$$\varphi \coloneqq R(s, A)\,(B\boldsymbol{u}\cdot\boldsymbol{n} + f_\varphi) \in L^2(\partial\Omega; V_0).$$

Using the property (61), we obtain that

$$A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n} = AR(s, A)\,(B\boldsymbol{u}\cdot\boldsymbol{n} + f_\varphi) + B\boldsymbol{u}\cdot\boldsymbol{n} \\ = sR(s, A)B\boldsymbol{u}\cdot\boldsymbol{n} + AR(s, A)f_\varphi$$

belongs to $L^2(\partial\Omega; V_1)$. We show that the IBC holds by rewriting (68) as

$$p = C(sR(s, A)B\boldsymbol{u}\cdot\boldsymbol{n} + AR(s, A)f_\varphi) \\ = C(AR(s, A)B\boldsymbol{u}\cdot\boldsymbol{n} + B\boldsymbol{u}\cdot\boldsymbol{n} + AR(s, A)f_\varphi) \\ = C(A\varphi + B\boldsymbol{u}\cdot\boldsymbol{n}),$$

using (64). Thus $(\boldsymbol{u}, p, \varphi) \in \mathcal{D}(\mathcal{A})$. The uniqueness of p follows from Theorem 11, that of $\boldsymbol{u}$ from (73), and that of φ from $s \in \rho(A)$. □

6. Addition of a derivative term.

By *derivative impedance* we mean

$$\hat{z}(s) = z_1 s,\ z_1 > 0,$$

for which the IBC (3) reduces to $p = z_1 \partial_t \boldsymbol{u} \cdot \boldsymbol{n}$.

The purpose of this section is to illustrate, on two examples, that the addition of such a derivative term to the IBCs covered so far (18,32,54) leaves unchanged the asymptotic stability results obtained with Corollary 8: it only makes the proofs more cumbersome as the state space becomes lengthier. This is why this term has not been included in Sections 3–5.

The examples will also illustrate why establishing the asymptotic stability of (2,3) with (4) can be done by treating each positive-real term in (4) separately (i.e. by building the realization of each of the four positive-real term separately and then aggregating them), thus justifying a posteriori the structure of the article.

Example 46 (Proportional-derivative impedance)**.** Consider the following positive-real impedance kernel

$$\hat{z}(s) = z_0 + z_1 s, \tag{69}$$

where $z_0, z_1 > 0$. The energy space is

$$H \coloneqq \nabla H^1(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega),$$
$$((\boldsymbol{u}, p, \eta), (\boldsymbol{f_u}, f_p, f_\eta))_H \coloneqq (\boldsymbol{u}, \boldsymbol{f_u}) + (p, f_p) + z_1(\eta, f_\eta)_{L^2(\partial\Omega)}$$

and the corresponding evolution operator is

$$\mathcal{D}(\mathcal{A}) \ni X \coloneqq \begin{pmatrix} \boldsymbol{u} \\ p \\ \eta \end{pmatrix} \longmapsto \mathcal{A}X \coloneqq \begin{pmatrix} -\nabla p \\ -\operatorname{div} \boldsymbol{u} \\ \frac{1}{z_1}\left[p - z_0 \boldsymbol{u} \cdot \boldsymbol{n}\right] \end{pmatrix},$$
$$\mathcal{D}(\mathcal{A}) \coloneqq \left\{ (\boldsymbol{u}, p, \eta) \in H \;\middle|\; \begin{array}{l} (\boldsymbol{u}, p) \in H_{\operatorname{div}}(\Omega) \times H^1(\Omega) \\ \eta = \boldsymbol{u} \cdot \boldsymbol{n} \text{ in } L^2(\partial\Omega) \end{array} \right\}.$$

Note how the derivative term in (69) is accounted for by adding the state variable $\eta \in L^2(\partial\Omega)$. The application of Corollary 8 is straightforward. For instance, for $X \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned} \Re(\mathcal{A}X, X)_H &= -\Re\left[(\boldsymbol{u} \cdot \boldsymbol{n}, p)_{L^2(\partial\Omega)}\right] + \Re\left[(p - z_0 \boldsymbol{u} \cdot \boldsymbol{n}, \eta)_{L^2(\partial\Omega)}\right] \\ &= -z_0 \|\boldsymbol{u} \cdot \boldsymbol{n}\|^2_{L^2(\partial\Omega)}, \end{aligned}$$

so that $\mathcal{A}$ is dissipative. The injectivity of $\mathcal{A}$ and the bijectivity of $s\mathcal{I} - \mathcal{A}$ for $s \in (0, \infty) \cup i\mathbb{R}^*$ can be proven similarly to what has been done in the previous sections.

Example 47. Let us revisit the delay impedance (18), covered in Section 3, by adding a derivative term to it:

$$\hat{z}(s) \coloneqq z_1 s + z_0 + z_\tau e^{-\tau s}, \tag{70}$$

where $z_1 > 0$ and (z_0, z_τ) are defined as in Section 3, so that $\hat{z}$ is positive-real. The inclusion of the derivative implies the presence of an additional variable in the extended state, i.e. the state space is (compare with (22))

$$H \coloneqq \nabla H^1(\Omega) \times L^2(\Omega) \times L^2(\partial\Omega; L^2(-\tau, 0)) \times L^2(\partial\Omega),$$
$$\begin{aligned} ((\boldsymbol{u}, p, \chi, \eta), (\boldsymbol{f_u}, f_p, f_\chi, f_\eta))_H \coloneqq (\boldsymbol{u}, \boldsymbol{f_u}) + (p, f_p) + k(\chi, f_\chi)_{L^2(\partial\Omega; L^2(-\tau,0))} \\ + z_1(\eta, f_\eta)_{L^2(\partial\Omega)}. \end{aligned}$$

The operator $\mathcal{A}$ becomes (compare with (23))

$$\mathcal{D}(\mathcal{A}) \ni X \coloneqq \begin{pmatrix} \boldsymbol{u} \\ p \\ \chi \\ \eta \end{pmatrix} \longmapsto \mathcal{A}X \coloneqq \begin{pmatrix} -\nabla p \\ -\operatorname{div} \boldsymbol{u} \\ \partial_\theta \chi \\ \frac{1}{z_1}\left[p - z_0 \boldsymbol{u}\cdot\boldsymbol{n} - z_\tau \chi(\cdot,-\tau)\right] \end{pmatrix},$$

$$\mathcal{D}(\mathcal{A}) \coloneqq \left\{ (\boldsymbol{u},p,\chi,\eta) \in H \;\middle|\; \begin{array}{l} (\boldsymbol{u},p,\chi) \in H_{\mathrm{div}}(\Omega) \times H^1(\Omega) \times L^2(\partial\Omega; H^1(-\tau,0)) \\ \chi(\cdot,0) = \boldsymbol{u}\cdot\boldsymbol{n} \text{ in } L^2(\partial\Omega) \\ \eta = \boldsymbol{u}\cdot\boldsymbol{n} \text{ in } L^2(\partial\Omega) \end{array} \right\},$$

where the IBC (3,70) is the third equation in $\mathcal{D}(\mathcal{A})$. The application of Corollary 8 is identical to Section 3.2. For instance, for $X \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned} \Re(\mathcal{A}X, X)_H &= -\Re\left[(\boldsymbol{u}\cdot\boldsymbol{n}, p)_{L^2(\partial\Omega)}\right] + \Re\left[k(\partial_\theta\chi,\chi)_{L^2(\partial\Omega;L^2(-\tau,0))}\right] \\ &\quad + \Re\left[(p - z_0\boldsymbol{u}\cdot\boldsymbol{n} - z_\tau\chi(\cdot,-\tau), \eta)_{L^2(\partial\Omega)}\right] \\ &= -\Re\left[(\boldsymbol{u}\cdot\boldsymbol{n}, p)_{L^2(\partial\Omega)}\right] + \frac{k}{2}\Re\left[\|\boldsymbol{u}\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)} - \|\chi(\cdot,-\tau)\|^2_{L^2(\partial\Omega)}\right] \\ &\quad + \Re\left[(p - z_0\boldsymbol{u}\cdot\boldsymbol{n} - z_\tau\chi(\cdot,-\tau), \boldsymbol{u}\cdot\boldsymbol{n})_{L^2(\partial\Omega)}\right] \\ &= \left(\frac{k}{2} - z_0\right)\|\boldsymbol{u}\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega)} - \frac{k}{2}\|\chi(\cdot,-\tau)\|^2_{L^2(\partial\Omega)} \\ &\quad - z_\tau\Re\left[(\chi(\cdot,-\tau), \boldsymbol{u}\cdot\boldsymbol{n})_{L^2(\partial\Omega)}\right], \end{aligned}$$

so that the expression of $\Re(\mathcal{A}X, X)_H$ is identical to that without a derivative term, see the proof of Lemma 23. The proof of the injectivity of $\mathcal{A}$ is also identical to that carried out in Lemma 24: the condition $\mathcal{A}X = 0$ yields $\chi(\cdot,0) = \chi(\cdot,-\tau) = \boldsymbol{u}\cdot\boldsymbol{n} = \eta$ a.e. on $\partial\Omega$. Finally, the proof of Lemma 25 can also be followed almost identically to solve $(s\mathcal{I} - \mathcal{A})X = F$ with $F = (\boldsymbol{f_u}, f_p, f_\chi, f_\eta)$, the additional steps being straightforward; after defining uniquely p, $\boldsymbol{u}$, and χ, the only possibility for η is $\eta \coloneqq \boldsymbol{u}\cdot\boldsymbol{n}$, which belongs to $L^2(\partial\Omega)$, and $\eta = \chi(\cdot,0)$ is deduced from (27).

7. Conclusions and perspectives.

This paper has focused on the asymptotic stability of the wave equation coupled with positive-real IBCs drawn from physical applications, namely time-delayed impedance in Section 3, standard diffusive impedance (e.g. fractional integral) in Section 4, and extended diffusive impedance (e.g. fractional derivative) in Section 5. Finally, the invariance of the derived asymptotic stability results under the addition of a derivative term in the impedance has been discussed in Section 6. The proofs crucially hinge upon the knowledge of a dissipative realization of the IBC, since it employs the semigroup asymptotic stability result given in [6, 40].

By combining these results, asymptotic stability is obtained for the impedance $\hat{z}$ introduced in Section 2 and given by (4). This suggests the first perspective of this work, formulated as a conjecture.

Conjecture 48. *Assume $\hat{z}$ is positive-real, without isolated singularities on $i\mathbb{R}$. Then the Cauchy problem (2,3) is asymptotically stable in a suitable energy space.*

Establishing this conjecture using the method of proof used in this paper first requires building a dissipative realization of the impedance operator $u \mapsto z \star u$.

If $\hat{z}$ is assumed rational and proper (i.e. $\hat{z}(\infty)$ is finite), a dissipative realization can be obtained using the celebrated positive-real lemma, also known as the

Kalman–Yakubovich–Popov lemma [5, Thm. 3]; the proof of asymptotic stability is then a simpler version of that carried out in Section 4, see [49, § 4.3] for the details. If $\hat{z}$ is not proper, it can be written as $\hat{z} = a_1 s + \hat{z}_{\mathrm{p}}$ where $a_1 > 0$ and $\hat{z}_{\mathrm{p}}$ is proper (see Remark 7); each term can be covered separately, see Section 6.

If $\hat{z}$ is not rational, then a suitable infinite-dimensional variant of the positive-real lemma is required. For instance, [61, Thm. 5.3] gives a realization using system nodes; a difficulty in using this result is that the properties needed for the method of proof presented here do not seem to be naturally obtained with system nodes. This result would be sharp, in the sense that it is known that exponential stability is not achieved in general (consider for instance $\hat{z}(s) = {}^{1}/_{\sqrt{s}}$ that induces an essential spectrum with accumulation point at 0). If this conjecture proves true, then the rate of decay of the solution could also be studied and linked to properties of the impedance $\hat{z}$; this could be done by adapting the techniques used in [63].

To illustrate this conjecture, let us give two examples of positive-real impedance kernels that are *not* covered by the results of this paper. Both examples arise in physical applications [48] and have been used in numerical simulations [47]. The first example is a kernel similar to (4), namely

$$\hat{z}(s) = z_0 + z_\tau e^{-\tau s} + z_1 s + \int_0^\infty \frac{\mu(\xi)}{s+\xi}\,\mathrm{d}\xi \quad (\Re(s) > 0)\,,$$

where $\tau > 0$, $z_\tau \in \mathbb{R}$, $z_0 \geq |z_\tau|$, $z_1 > 0$, and the weight $\mu \in \mathcal{C}^\infty((0,\infty))$ satisfies the condition $\int_0^\infty \frac{|\mu(\xi)|}{1+\xi}\mathrm{d}\xi < \infty$ and is such that $\hat{z}$ is positive-real. When the sign of μ is indefinite the passivity condition (44) does not hold, so that this impedance is not covered by the presented results despite the fact that, overall, $\hat{z}$ is positive-real with a realization formally identical to that of the impedance (4) defined in Section 2.

The second and last example is

$$\hat{z}(s) = z_0 + z_\tau \frac{e^{-\tau s}}{\sqrt{s}},$$

with $z_\tau \geq 0$, $\tau > 0$, and $z_0 \geq 0$ sufficiently large for $\hat{z}$ to be positive-real (the precise condition is $z_0 \geq -z_\tau \cos(\tilde{x} + \frac{\pi}{4})\sqrt{\tau/\tilde{x}}$ where $\tilde{x} \simeq 2.13$ is the smallest positive root of $x \mapsto \tan(x + {}^{\pi}/_{4}) + {}^{1}/_{2x}$). A simple realization can be obtained by combining Sections 3 and 4, i.e. by delaying the diffusive representation using a transport equation: the convolution then reads, for a causal input u,

$$z \star u = z_0 u + z_\tau \int_0^\infty \chi(t, -\tau, \xi)\,\mathrm{d}\mu(\xi),$$

where φ and μ are defined as in Section 4, and for a.e. $\xi \in (0,\infty)$ the function $\chi(\cdot,\cdot,\xi)$ obeys the transport equation (21ab) but with $\chi(t,0,\xi) = \varphi(t,\xi)$. So far, the authors have not been able to find a suitable Lyapunov functional (i.e. a suitable definition of $\|\cdot\|_H$) for this realization.

The second open problem we wish to point out is the extension of the stability result to discontinuous IBCs. A typical case is a split of the boundary $\partial\Omega$ into three disjoint parts: a Neumann part $\partial\Omega_N$, a Dirichlet part $\partial\Omega_D$, and an impedance part $\partial\Omega_z$ where one of the IBCs covered in the paper is applied. Dealing with such discontinuities may involve the redefinition of both the energy space H and domain $\mathcal{D}(\mathcal{A})$, as well as the derivation of compatibility constraints. The proofs may benefit from considering the scattering formulation, recalled in Remark 2, which enables to write the three boundary conditions in a unified fashion.

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Appendix A. **Miscellaneous results.** For the sake of completeness, the key technical results upon which the paper depends are briefly gathered here.

A.1. **Extension by zero.** Let us define the zero extension operator as

$$E:\, L^2(\Omega_1) \to L^2(\Omega_2),\ Eu \coloneqq \{u \text{ on } \Omega_1,\ 0 \text{ on } \Omega_2\backslash\Omega_1,$$

where Ω_1 and Ω_2 are two open subsets of $\mathbb{R}^d$ such that $\overline{\Omega_1} \subset \Omega_2$.

Proposition 49. *Let Ω_1 and Ω_2 be two bounded open subsets of $\mathbb{R}^d$ such that $\overline{\Omega_1} \subset \Omega_2$. For any $p \in H_0^1(\Omega_1)$, $Ep \in H_0^1(\Omega_2)$ with*

$$\forall i \in [\![1, d]\!],\ \partial_i [Ep] = E[\partial_i p] \ \text{ a.e. in } \ \Omega_2. \tag{71}$$

In particular, $\|p\|_{H^1(\Omega_1)} = \|Ep\|_{H^1(\Omega_2)}$.

Remark. Note that we do not require any regularity on the boundary of Ω_i. This is due to the fact that the proof only relies on the definition of H_0^1 by density.

Proof. The first part of the proof is adapted from [3, Lem. 3.22]. By definition of $H_0^1(\Omega_1)$, there is a sequence $\phi_n \in \mathcal{C}_0^\infty(\Omega_1)$ converging to p in the $\|\cdot\|_{H^1(\Omega_1)}$ norm. Since $Ep \in L^2(\Omega_2)$, Ep is locally integrable and thus belongs to $\mathcal{D}'(\Omega_2)$. For any $\varphi \in \mathcal{C}_0^\infty(\Omega_2)$, we have

$$\begin{aligned}
\langle \partial_i [Ep], \varphi\rangle_{\mathcal{D}'(\Omega_2),\mathcal{C}_0^\infty(\Omega_2)} &\coloneqq -\langle Ep, \partial_i\varphi\rangle_{\mathcal{D}'(\Omega_2),\mathcal{C}_0^\infty(\Omega_2)} \\
&= -\int_{\Omega_1} p\partial_i\varphi && \big(Ep \in L^2(\Omega_2)\big) \\
&= -\lim_{n\to\infty}\int_{\Omega_1} \phi_n\partial_i\varphi && \big(p \in H_0^1(\Omega_1)\big) \\
&= \lim_{n\to\infty}\int_{\Omega_1} \partial_i\phi_n\varphi && \big(\phi_n \in \mathcal{C}_0^\infty(\Omega_1)\big) \\
&= \int_{\Omega_1} \partial_i p\varphi && \left(\partial_i\phi_n \xrightarrow[n\to\infty]{L^2(\Omega_1)} \partial_i p\right) \\
&= \int_{\Omega_2} E[\partial_i p]\varphi,
\end{aligned}$$

hence $E[\partial_i p] = \partial_i[Ep]$ in $\mathcal{D}'(\Omega_2)$. Since $\partial_i p \in L^2(\Omega_1)$ by assumption, we deduce from this identity that $E[\partial_i p] \in L^2(\Omega_2)$. Hence $Ep \in H^1(\Omega_2)$.

Using the fact that E is an isometry from $H_0^1(\Omega_1)$ to $H^1(\Omega_2)$ we deduce

$$\|E\phi_n - Ep\|_{H^1(\Omega_2)} = \|E(\phi_n - p)\|_{H^1(\Omega_2)} = \|\phi_n - p\|_{H^1(\Omega_1)} \xrightarrow[n\to\infty]{} 0.$$

Since $E\phi_n \in \mathcal{C}_0^\infty(\Omega_2)$, this shows that $Ep \in H_0^1(\Omega_2)$. □

A.2. **Compact embedding and trace operator.** Let $\Omega \subset \mathbb{R}^d$, $d \in [\![1,\infty[\![$, be a bounded open set with a Lipschitz boundary.

The embedding $H^1(\Omega) \subset H^s(\Omega)$ with $s \in [0,1)$ is compact [26, Thm. 1.4.3.2]. (See [36, Thm. 16.17] for smooth domains.)

The trace operator $H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ with $s \in (1/2, 1]$ is continuous and surjective [26, Thm. 1.5.1.2]. (See [18, Thm. 1] if Ω is also simply connected and [36, Thm. 9.4] for smooth domains.)

The trace operator $H_{\text{div}}(\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$, $\boldsymbol{u} \mapsto \boldsymbol{u}\cdot\boldsymbol{n}$ is continuous [23, Thm. 2.5], and the following Green's formula holds for $\psi \in H^1(\Omega)$ [23, Eq. (2.17)]

$$(\boldsymbol{u}, \nabla\psi) + (\operatorname{div}\boldsymbol{u}, \psi) = \langle \boldsymbol{u}\cdot\boldsymbol{n}, \overline{\psi}\rangle_{H^{-\frac{1}{2}}(\partial\Omega), H^{\frac{1}{2}}(\partial\Omega)}. \tag{72}$$

A.3. **Hodge decomposition.** Let $\Omega \subset \mathbb{R}^d$, $d \in [\![1,\infty[\![$, be a connected open set with a Lipschitz boundary. The following orthogonal decomposition holds [15, Prop. IX.1]

$$(L^2(\Omega))^d = \nabla H^1(\Omega) \oplus H_{\text{div}\,0,0}(\Omega), \tag{73}$$

where

$$\nabla H^1(\Omega) \coloneqq \left\{\boldsymbol{f} \in (L^2(\Omega))^d \mid \exists g \in H^1(\Omega) : \ \boldsymbol{f} = \nabla g\right\}$$

is a closed subspace of $(L^2(\Omega))^d$ and

$$H_{\text{div}\,0,0}(\Omega) \coloneqq \left\{\boldsymbol{f} \in H_{\text{div}}(\Omega) \mid \operatorname{div}\boldsymbol{f} = 0,\ \boldsymbol{f}\cdot\boldsymbol{n} = 0 \text{ in } H^{-\frac{1}{2}}(\partial\Omega)\right\}.$$

This result still holds true when Ω is disconnected (the proof of [15, Prop. IX.1] relies on Green's formula (72) as well as the compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$, needed to apply Peetre's lemma).

Remark 50. The space $H_{\text{div}\,0,0}(\Omega)$ is studied in [15, Chap. IX] for $n = 2$ or 3. For instance,

$$\mathbb{H}_1 \coloneqq H_{\text{div}\,0,0}(\Omega) \cap \left\{\boldsymbol{f} \in (L^2(\Omega))^d \mid \nabla \times \boldsymbol{f} = \boldsymbol{0}\right\}$$

has a finite dimension under suitable assumptions on the set Ω [15, Prop. IX.2].

A.4. **Semigroups of linear operators.**

Theorem 51 (Lumer-Phillips)**.** *Let H be a complex Hilbert space and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \to H$ an unbounded operator. If $\Re(\mathcal{A}X, X)_H \leq 0$ for every $X \in \mathcal{D}(\mathcal{A})$ and $\mathcal{I} - \mathcal{A}$ is surjective, then $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup of contractions $\mathcal{T}(t) \in \mathcal{L}(H)$.*

Proof. The result follows from [52, Thms. 4.3 & 4.6] since Hilbert spaces are reflexive [34, Thm. 8.9]. □

Theorem 52 (Asymptotic stability [6, 40])**.** *Let H be a complex Hilbert space and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H \to H$ be the infinitesimal generator of a strongly continuous semigroup $\mathcal{T}(t) \in \mathcal{L}(H)$ of contractions. If $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \varnothing$ and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, then $\mathcal{T}$ is asymptotically stable, i.e. $\mathcal{T}(t)X_0 \to 0$ in H as $t \to \infty$ for any $X_0 \in H$.*

Appendix B. **Application of the invariance principle.** The purpose of this appendix is to justify why, in this paper, we rely on Corollary 8 rather than the invariance principle, commonly used with dynamical systems on Banach spaces. Theorem 53 below states the invariance principle for the case of interest herein, i.e. a linear Cauchy problem (8) for which the Lyapunov functional is $\frac{1}{2}\|\cdot\|_H^2$. (For further background, see [39, § 3.7] and [9, Chap. 9].)

Theorem 53 (Invariance principle)**.** *Let $\mathcal{A}$ be the infinitesimal generator of a strongly continuous semigroup of contractions $\mathcal{T}(t) \in \mathcal{L}(H)$ and $X_0 \in H$. If the orbit $\gamma(X_0) \coloneqq \bigcup_{t\geq 0} \mathcal{T}(t)X_0$ lies in a compact set of H, then $\mathcal{T}(t)X_0 \to M$ as $t \to \infty$, where M is the largest $\mathcal{T}$-invariant set in*

$$\{X \in \mathcal{D}(\mathcal{A}) \mid \Re\left[(\mathcal{A}X, X)_H\right] = 0\}. \tag{74}$$

Proof. The function $\Phi \coloneqq \frac{1}{2}\|\cdot\|_H^2$ is continuous on H and satisfies $\Phi(\mathcal{T}(t)X) \leq \Phi(X)$ for any $X \in H$ so that it is a Lyapunov functional. The invariance principle [27, Thm. 1] then shows that $\mathcal{T}(t)X_0$ is attracted to the largest invariant set of

$$\left\{X \in H \;\middle|\; \lim_{t\to 0^+} t^{-1}(\Phi(\mathcal{T}(t)X) - \Phi(X)) = 0\right\}.$$

□

Let us now discuss the application of Theorem 53 to (2,3), assuming we know a dissipative realization of the impedance operator $u \mapsto z \star u$ in a state space V_0.

The first step is to establish that the largest invariant subset of (74) reduces to $\{0\}$, i.e. that the only solution of (8) in (74) is null, which is verified by the evolution operators defined in Sections 3–6. This requires to exclude solenoidal fields from X_0, see Remark 22.

The second step is to prove the precompactness of the orbit $\gamma(X_0)$ for any X_0 in H. The following criterion can be used, where for $s \in \rho(\mathcal{A})$ we denote the resolvent operator by

$$R(s, \mathcal{A}) \coloneqq (s\mathcal{I} - \mathcal{A})^{-1}. \tag{75}$$

Theorem 54 ([14, Thm. 3])**.** *Let $\mathcal{A}$ be the infinitesimal generator of a strongly continuous semigroup of contractions on H. If $R(s, \mathcal{A})$ is compact for some $s > 0$, then $\gamma(X_0)$ is precompact for any $X_0 \in H$.*

Using Theorem 54 reduces to proving that the embedding $\mathcal{D}(\mathcal{A}) \subset H$ is compact, which based on the examples covered in this paper boils down to proving that the embeddings

$$H_{\mathrm{div}}(\Omega) \times H^1(\Omega) \subset \nabla H^1(\Omega) \times L^2(\Omega) \quad (\mathrm{a}), \; L^2(\partial\Omega; V_1) \subset L^2(\partial\Omega; V_0) \quad (\mathrm{b}) \tag{76}$$

are compact, where V_0 is the energy space of the extended variables and $V_1 \subset V_0$.

The compactness of the embedding (76a) is obvious if $d = 1$. If $d = 3$, it can be proven using the following regularity result: if Ω is a bounded simply connected open set with Lipschitz boundary, [12, Thm. 2]

$$H_{\mathrm{curl}}(\Omega) \cap \left\{\boldsymbol{u} \in H_{\mathrm{div}}(\Omega) \;\middle|\; \boldsymbol{u} \cdot \boldsymbol{n} \in L^2(\partial\Omega)\right\} \subset H^{\frac{1}{2}}(\Omega)^d$$

and $\nabla H^1(\Omega) \subset H_{\mathrm{curl}}(\Omega)$ [23, Thm. 2.9]. (Note the stringent requirement that Ω be simply connected.)

The compactness of (76b) depends upon both d and the impedance kernel z. If $d = 1$, then it holds true if $V_1 \subset V_0$ is compact (which is satisfied by the delay impedance covered in Section 3, where $V_1 = H^1(-\tau, 0)$ and $V_0 = L^2(-\tau, 0)$, but not by the diffusive impedances covered in Sections 4–5) or if both V_1 and V_0 are finite-dimensional (which is verified for a rational impedance). If $d > 1$, then it is not obvious.

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