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ORIGINAL ARTICLE

Recovering the observable part of the initial data of an infinite-dimensional linear system with skew-adjoint generator

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Abstract We consider the problem of recovering the initial data (or initial state) of infinite-dimensional linear systems with unitary semigroups. It is well-known that this inverse problem is well posed if the system is exactly observable, but this assumption may be very restrictive in some applications. In this paper we are interested in systems which are not exactly observable, and in particular, where we cannot expect a full reconstruction. We propose to use the algorithm studied by Ramdani et al. in (Automatica 46:1616–1625, 2010) and prove that it always converges towards the observable part of the initial state. We give necessary and sufficient condition to have an exponential rate of convergence. Numerical simulations are presented to illustrate the theoretical results.

Keywords Linear systems · Inverse problems · Controllability · Observability · Feedback control

1 Introduction

1.1 Motivation

In many areas of science, we need to recover the initial (or final) data of a physical system from partial observation over some finite time interval. In oceanography and

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meteorology, where this problem is known as *data assimilation*, we can mention the works of Auroux and Blum [1–3], Gejadze et al. [19,27], Shutyaev and Gejadze [34], Teng et al. [38] and the monograph of Blum et al. [7] concerning the numerical aspects. This problem also arises in medical imaging, for instance in thermoacoustic tomography. There, the problem is to recover the initial data of a wave type equation from surface measurements (see Gebauer and Scherzer [18] and the survey of Kuchment and Kunyansky [26]).

In the last decade, new algorithms based on time reversal (see Fink [15,16]) have been proposed for this problem. We can mention, for instance, the Back and Forth Nudging proposed by Auroux and Blum [1], the Time Reversal Focusing by Phung and Zhang [31], the algorithm proposed by Ito et al. [24] and finally, the one we will consider in this paper, the forward–backward observers-based algorithm proposed by Ramdani et al. [32] (which is a generalization of the one in [31]). In this paper, we study the convergence of the reconstruction algorithm of [32] for systems with skew-adjoint generator, when the inverse problem is ill-posed, that is to say when either the observability or the estimatability assumption fails.

To make this statement precise, let us begin with some notation and definitions. Let X be a Hilbert space and A a skew-adjoint operator on X. We are interested in the reconstruction of the initial data z_0 of

$$\begin{cases} \dot{z}(t) = Az(t) \\ z(0) = z_0 \in X. \end{cases} \quad \forall\, t \geq 0, \tag{1.1}$$

Such equations are often used to model vibrating systems (acoustic or elastic waves) or quantum systems (Schrödinger equations).

By Stone's Theorem (see for instance Tucsnak and Weiss [39]), A is the infinitesimal generator of a unitary C_0-group $\mathbb{S}$ on X, and in particular, $\|z(t)\| = \|z_0\|$ for all $t \geq 0$.

Let Y be another Hilbert space. We suppose that we have access to z through the operator $C : \mathcal{D}(A) \to Y$, during a time interval $[0, \tau]$, $\tau > 0$, leading to the measurement

$$y(t) = Cz(t) \quad \forall\, t \in [0, \tau]. \tag{1.2}$$

We call C the observation operator of the system. The observation is said to be *bounded* if C is a bounded operator (i.e. $C \in \mathcal{L}(X, Y)$), and unbounded otherwise. In the latter case, we still assume that C is bounded with respect to the graph norm of A on $\mathcal{D}(A)$.

For systems described by evolution partial differential equations (i.e. when A is a differential operator in the space variables on a domain Ω), bounded observation generally corresponds to measurement on a subdomain $\mathcal{O} \subset \Omega$, while unbounded observation in most cases corresponds to measurement on the boundary of Ω.

If we denote Ψ_τ the operator which associates the output function $y|_{[0,\tau]}$ to an initial data $z_0 \in \mathcal{D}(A)$, the inverse problem is well posed when Ψ_τ is left-invertible, with bounded left-inverse. This is equivalent to Ψ_τ being bounded from below

$$\exists k_\tau > 0, \quad \|\Psi_\tau z_0\| \geq k_\tau \|z_0\| \quad \forall\, z_0 \in \mathcal{D}(A). \tag{1.3}$$

The pair (A, C) is said to be *exactly observable in time* τ when (1.3) holds.

Now, we present the algorithm proposed by Ramdani et al. [32]. For simplicity, we consider the particular case where A is skew-adjoint and $C \in \mathcal{L}(X, Y)$, the pair (A, C) being exactly observable in time $\tau > 0$. Let $\mathbb{T}^+$ be the exponentially stable C_0-semigroup generated by $A^+ = A - \gamma C^*C$, while $\mathbb{T}^-$ is generated by $A^- = -A - \gamma C^*C$, for some $\gamma > 0$ (see Liu [28]). For all $n \in \mathbb{N}^*$, we define the following systems

$$\begin{cases} \dot{z}_n^+(t) = A^+ z_n^+(t) + \gamma C^* y(t) & \forall\, t \in [0, \tau], \\ z_1^+(0) = z_0^+ \in X, \\ z_n^+(0) = z_{n-1}^-(0) & \forall\, n \geq 2, \end{cases} \tag{1.4}$$

$$\begin{cases} \dot{z}_n^-(t) = -A^- z_n^-(t) - \gamma C^* y(t) & \forall\, t \in [0, \tau], \\ z_n^-(\tau) = z_n^+(\tau) & \forall\, n \geq 1. \end{cases} \tag{1.5}$$

The forward error $e_n^+(t) = z_n^+(t) - z(t)$ satisfies

$$\begin{cases} \dot{e}_n^+(t) = (A - \gamma C^*C) e_n^+(t) & \forall\, t \in [0, \tau], \\ e_1^+(0) = z_0^+ - z_0 \in X, \\ e_n^+(0) = e_{n-1}^-(0) & \forall\, n \geq 2, \end{cases}$$

and the backward error $e_n^-(t) = z_n^-(t) - z(t)$

$$\begin{cases} \dot{e}_n^-(t) = (A + \gamma C^*C) e_n^-(t) & \forall\, t \in [0, \tau], \\ e_n^-(\tau) = e_n^+(\tau) & \forall\, n \geq 1. \end{cases}$$

So, we have

$$\left\| z_n^-(0) - z_0 \right\| = \left\| e_n^-(0) \right\| = \left\| \left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+ \right)^n e_1^+(0) \right\| \leq \left\| \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \right\|^n \left\| z_0^+ - z_0 \right\|. \tag{1.6}$$

According to Ito et al. [24, Lemma 2.2], if (A, C) is exactly observable in time τ, we have $\left\| \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \right\|_{\mathcal{L}(X)} = \alpha < 1$ and thus

$$\| z_n^-(0) - z_0 \| \leq \alpha^n \| z_0^+ - z_0 \| \underset{n \to \infty}{\longrightarrow} 0.$$

In the case of exactly observable systems, we call the systems (1.4)–(1.5) *forward* and *backward observers* as it is a generalization to infinite-dimensional systems of the so-called Luenberger's observers [29], well-known in control theory. Observers for infinite-dimensional systems are an active topic of research, for both linear or non-linear systems, and among the large literature, we can cite for instance: Chapelle et al. [9], Krstic et al. [25], Moireau et al. [30], Smyshlyaev and Krstic [35], and Couchouron and Ligarius [10]. For pioneering work, we refer to Baras and Bensoussan [4] and Bensoussan [6].

In the paper of Ramdani et al. [32], they consider a wide class of infinite-dimensional systems (allowing even an observation operator that is not admissible). They suppose that the system is estimatable and backward estimatable (roughly speaking, the system can be forward and backward stabilized with a feedback operator called a *stabilizing output injection operator*). However, they show in Proposition 3.3 that this implies that the system is exactly observable, or in other words, that (1.3) is satisfied (for some sufficiently large time τ). In this paper, we are dealing with the initial data recovery of some well posed linear systems which are not supposed to be exactly observable, using the same algorithm.

By a well posed linear system we mean a linear time-invariant system Σ such that on any finite time interval $[0, t]$, the operator Σ_t from the initial state z_0 and the input function u to the final state $z(t)$ and the output function y is bounded. In other words, Σ is a family of bounded operators such that

$$\begin{bmatrix} z(t) \\ y|_{[0,t]} \end{bmatrix} = \Sigma_t \begin{bmatrix} z_0 \\ u|_{[0,t]} \end{bmatrix}.$$

Under some assumptions on the system Σ, we propose to investigate the above algorithm in the framework of well posed linear systems (allowing admissible observation operators) to recover the *observable part* of z_0 from $y|_{[0,\tau]}$. The results on well posed linear systems used in this work will be recalled in Sect. 2. For more details, we refer the reader, for instance, to the work of Salamon et al. [33,36,37,40–42] and the survey of Weiss et al. [45].

The paper is organized as follows. In Sect. 2 we give some background on well posed linear systems, including the construction of the dual system and the known results on colocated feedback. In Sect. 3, we begin with the definition of two systems, Σ^+ and Σ^-, corresponding to the forward (1.4) and backward (1.5) observers, respectively. We then work on the properties of the operator $\mathbb{T}_\tau^- \mathbb{T}_\tau^+$, called the *forward–backward operator*, which appears naturally. The properties of this operator, given in Proposition 3.9, are needed to prove the main result of this paper. Finally, we prove the main result of this work, Theorem 1.1, which shows that the algorithm leads to the reconstruction of the observable part of the initial state. In Sect. 4, we apply our theoretical result to an N-dimensional ($N \geq 2$) wave equation, with Dirichlet control and colocated observation on a part of the boundary.

1.2 Main results

From a well posed linear system $\Sigma = \begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix}$, defined in Definition 2.1 and verifying some assumptions (namely $A^* = -A$ and $B = C^*$), we will construct two other well posed linear systems Σ^+ and Σ^-, corresponding to (1.4) and (1.5), respectively. All the needed terminology and results on well posed linear systems are recalled in Sect. 2.

Let us begin with the definition of the *time-reflection operator*. Let W be a Hilbert space. For all $\tau \geq 0$, we define the linear operator $\text{Я}_\tau : L^2_{\ell oc}([0, \infty), W) \to L^2_{\ell oc}([0, \infty), W)$ by

$$(\mathcal{R}_\tau u)(t) = \begin{cases} u(\tau - t) & \forall\, t \in [0, \tau], \\ 0 & \forall\, t > \tau. \end{cases}$$

To state our main result, we need the operator Φ_τ^d defined in Theorem 2.13. In the following theorem, we only need that $\Phi_\tau^d = \Psi_\tau^* \mathcal{R}_\tau$, so that $V_{\mathrm{Obs}} = \overline{\mathrm{Ran}\, \Phi_\tau^d}$ can be understood as $(\mathrm{Ker}\, \Psi_\tau)^\perp$. From that, the link with the known results in the case of exact observability (1.3) is obvious.

Theorem 1.1 *Let X and Y be Hilbert spaces. Assume that Σ is a well posed linear system with input and output space Y and state space X determined by the operators (A, B, C) and the transfer function* **G***, such that $A^* = -A$ and $B = C^*$. Using Theorems 2.13 and 2.17, let us denote by Σ^+ (resp. Σ^-) the closed-loop system of Σ (resp. Σ^d) with output feedback operator γI, where $\gamma \in (0, \kappa)$, for some $\kappa \in (0, \infty]$ (explicitly given in Remark 2.18).*

Let $z_0 \in X$ and denote u, z and y the input, trajectory and output of Σ, respectively, with initial state z_0. Let $\tau > 0$, $z_0^+ \in X$ and denote, for all $n \geq 1$, z_n^+ and z_n^- the respective trajectories of Σ^+ and Σ^- with respective inputs $v^+ = \gamma y + u$ and $v^- = \gamma\, \mathcal{R}_\tau\, y + \mathcal{R}_\tau\, u$, and initial states

$$z_1^+(0) = z_0^+ \in X, \quad z_n^+(0) = z_{n-1}^-(0), \ n \geq 2, \quad z_n^-(\tau) = z_n^+(\tau), \ n \geq 1.$$

Furthermore, we denote by Π the orthogonal projector from X onto $V_{\mathrm{Obs}} = \overline{\mathrm{Ran}\, \Phi_\tau^d}$, then the following statements hold true:

1. *We have for all $z_0, z_0^+ \in X$*

$$\left\| (I - \Pi) \left(z_n^-(0) - z_0 \right) \right\| = \left\| (I - \Pi) \left(z_0^+ - z_0 \right) \right\| \quad \forall\, n \geq 1.$$

2. *The sequence $\left(\left\| \Pi \left(z_n^-(0) - z_0 \right) \right\| \right)_{n \geq 1}$ is strictly decreasing and satisfies*

$$\left\| \Pi \left(z_n^-(0) - z_0 \right) \right\| \underset{n \to \infty}{\longrightarrow} 0.$$

3. *The rate of convergence is exponential,* i.e. there exists a constant $\alpha \in (0, 1)$, independent of z_0 and z_0^+, such that

$$\left\| \Pi \left(z_n^-(0) - z_0 \right) \right\| \leq \alpha^n \left\| \Pi \left(z_0^+ - z_0 \right) \right\| \quad \forall\, n \geq 1,$$

   if and only if $\mathrm{Ran}\, \Phi_\tau^d$ is closed in X.

Theorem 1.1 allows us to approximate the projection of z_0 on V_{Obs} by the projection of $z_n^-(0)$. However, in practice, it is difficult to characterize V_{Obs} and thus the projector Π. The following corollary shows that if the (arbitrary) initial guess z_0^+ belongs to V_{Obs} (for example, one can take $z_0^+ = 0$), then all successive approximations $z_n^-(0)$ belong to V_{Obs}, so that we do not need to know Π anymore.

Corollary 1.2 *Under the assumptions of Theorem 1.1, if $z_0^+ \in V_{\text{Obs}}$, then*

$$\left\| z_n^-(0) - \Pi z_0 \right\| \underset{n\to\infty}{\longrightarrow} 0.$$

Furthermore, the decay rate is exponential if and only if $\operatorname{Ran} \Phi_\tau^d$ *is closed in X.*

We will prove this corollary in Sect. 3.4.

2 Background on well posed linear systems

In this section, we recall some definitions used in the framework of *well posed linear systems*, also called *abstract linear systems*. All this material can be found, for instance, in [33,36,37,40–42,45].

2.1 Definitions and associated operators A, B and C

We first define the τ*-concatenation*. For any $\tau \geq 0$ and any Z, Hilbert space, we define for all u, v in $L^2([0,\infty), Z)$ the following binary operator

$$(u \diamond_\tau v)(t) = \begin{cases} u(t) & \forall\, t \in [0,\tau), \\ v(t-\tau) & \forall\, t \geq \tau. \end{cases}$$

Definition 2.1 (*Well posed linear system*) Let X, U and Y be Hilbert spaces. We denote by $\mathcal{U} = L^2([0,\infty), U)$ and $\mathcal{Y} = L^2([0,\infty), Y)$. A *well posed linear system* on $(\mathcal{U}, X, \mathcal{Y})$ is a family of bounded operators $\Sigma = (\Sigma_t)_{t\geq 0}$ from $X \times \mathcal{U}$ to $X \times \mathcal{Y}$, where $\Sigma_t = \begin{bmatrix} \mathbb{T}_t & \Phi_t \\ \Psi_t & \mathbb{F}_t \end{bmatrix}$, satisfying:

- $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ is a C_0-semigroup on X,
- $\Phi = (\Phi_t)_{t\geq 0}$ is a family of bounded linear operators from $\mathcal{U}$ to X such that

$$\Phi_{\tau+t}(u \diamond_\tau v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v \quad \forall\, u, v \in \mathcal{U},\ \tau, t \geq 0,$$

- $\Psi = (\Psi_t)_{t\geq 0}$ is a family of bounded linear operators from X to $\mathcal{Y}$ such that

$$\Psi_{\tau+t} z = \Psi_\tau z \diamond_\tau \Psi_t \mathbb{T}_\tau z \quad \forall\, z \in X,\ \tau, t \geq 0,$$

  and $\Psi_0 \equiv 0$,
- $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$ such that

$$\mathbb{F}_{\tau+t}(u \diamond_\tau v) = (\mathbb{F}_\tau u) \diamond_\tau (\Psi_t \Phi_\tau u + \mathbb{F}_t v) \quad \forall\, u, v \in \mathcal{U},\ \tau, t \geq 0,$$

  and $\mathbb{F}_0 \equiv 0$.

We call U the *input space* of Σ, X the *state space* of Σ, and Y the *output space* of Σ. The operator Φ_τ is called an *input map*, Ψ_τ an *output map* and $\mathbb{F}_\tau$ an *input–output map*.

Denoting by $\mathbf{P}_\tau$ the projection of $L^2([0,\infty), Z)$ on $L^2([0,\tau), Z)$ (by truncation), one can easily show that $\Phi_\tau \mathbf{P}_\tau = \Phi_\tau$, $\mathbb{F}_\tau \mathbf{P}_\tau = \mathbb{F}_\tau$, $\mathbf{P}_t \Psi_\tau = \Psi_t$ and $\mathbf{P}_t \mathbb{F}_\tau \mathbf{P}_t = \mathbf{P}_t \mathbb{F}_\tau = \mathbb{F}_t$ for all $0 \leq t \leq \tau$.

To be able to define the output y of the system Σ from its operators, we first need to define

$$\Psi_\infty = \lim_{\tau\to\infty} \Psi_\tau \in \mathcal{L}(X, \mathcal{Y}_{\ell oc}),$$

and

$$\mathbb{F}_\infty = \lim_{\tau\to\infty} \mathbb{F}_\tau \in \mathcal{L}(\mathcal{U}_{\ell oc}, \mathcal{Y}_{\ell oc}),$$

where $\mathcal{U}_{\ell oc}$ and $\mathcal{Y}_{\ell oc}$ are the Fréchet spaces defined by $\mathcal{U}_{\ell oc} = L^2_{\ell oc}([0,\infty), U)$ and $\mathcal{Y}_{\ell oc} = L^2_{\ell oc}([0,\infty), Y)$ with the seminorms being the norms of $\mathbf{P}_\tau u$, where $\tau > 0$. Then, one can easily show that

$$\Psi_\tau = \mathbf{P}_\tau \Psi_\infty, \quad \mathbb{F}_\tau = \mathbf{P}_\tau \mathbb{F}_\infty.$$

We call Ψ_∞ an *extended output map* of Σ, and $\mathbb{F}_\infty$ an *extended input–output map* of Σ.

Definition 2.2 Let $z_0 \in X$ and $u \in \mathcal{U}_{\ell oc}$, the *state trajectory* z and the *output function* y of Σ corresponding to the initial state z_0 and the input function u are defined by

$$\begin{aligned} z(t) &= \mathbb{T}_t z_0 + \Phi_t u \quad \forall\, t \geq 0, \\ y &= \Psi_\infty z_0 + \mathbb{F}_\infty u. \end{aligned} \tag{2.1}$$

One can easily see that

$$\begin{bmatrix} z & (t) \\ \mathbf{P}_t & y \end{bmatrix} = \Sigma_t \begin{bmatrix} z_0 \\ \mathbf{P}_t \quad u \end{bmatrix}.$$

Let A be the infinitesimal generator of $\mathbb{T}$, and $\omega_0(\mathbb{T})$ its *growth bound*. We denote by X_1 the domain $\mathcal{D}(A)$ endowed with the graph norm, denoting by $\|\cdot\|_1$, and X_{-1} the closure of X with the norm $\|z\|_{-1} = \|(\beta I - A)^{-1} z\|$ (for some arbitrary $\beta \in \rho(A)$, the resolvent set of A). It is well-known (see for instance Tucsnak and Weiss [39]) that these spaces are Hilbert spaces and that

$$X_1 \subset X \subset X_{-1},$$

each inclusion being dense and with continuous embedding.

For any Hilbert space W, any interval J and any $\omega \in \mathbb{R}$, we denote by

$$L^2_\omega(J, W) = e_\omega L^2(J, W),$$

where $(e_\omega v)(t) = e^{\omega t} v(t)$, with the norm $\|e_\omega v\|_{L^2_\omega} = \|v\|_{L^2}$.

Proposition 2.3 *There exists a unique operator $B \in \mathcal{L}(U, X_{-1})$, called the control operator of Σ, such that for any initial state $z_0 \in X$ and any input function $u \in \mathcal{U}_{\ell oc}$, the state trajectory z defined in (2.1) is the unique strong solution in X_{-1} of*

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) & \forall\, t \geq 0, \\ z(0) = z_0. \end{cases}$$

Moreover, we know that $z \in C([0,\infty), X) \cap H^1_{\ell oc}([0,\infty), X_{-1})$, and if $u \in L^2_\omega([0,\infty), U)$ with $\omega > \omega_0(\mathbb{T})$, then z also belongs to $L^2_\omega([0,\infty), X)$ and its Laplace transform is

$$\widehat{z}(s) = (sI - A)^{-1}[z_0 + B\widehat{u}(s)] \quad \forall\, s \in \mathbb{C}_\omega.$$

We can also prove that

$$\Psi_\infty \in \mathcal{L}(X, L^2_\omega([0,\infty), Y)),$$

and

$$\mathbb{F}_\infty \in \mathcal{L}\left(L^2_\omega([0,\infty), U), L^2_\omega([0,\infty), Y)\right).$$

This enables us to represent y *via* its Laplace transform.

Proposition 2.4 *There exist an analytic $\mathcal{L}(U, Y)$-valued function $\mathbf{G}$ on $\mathbb{C}_{\omega_0(\mathbb{T})}$, called the transfer function of Σ, and a unique operator $C \in \mathcal{L}(X_1, Y)$, called the observation operator of Σ, with the following properties:*

- *For every $z_0 \in X$ and $u \in L^2_\omega([0,\infty), U)$ with $\omega > \omega_0(\mathbb{T})$, the corresponding output function $y = \Psi_\infty z_0 + \mathbb{F}_\infty u$ belongs to $L^2_\omega([0,\infty), Y)$ and its Laplace transform is*
$$\widehat{y}(s) = C(sI - A)^{-1} z_0 + \mathbf{G}(s)\widehat{u}(s) \quad \forall\, s \in \mathbb{C}_\omega. \tag{2.2}$$
- *$\mathbf{G}$ satisfies for all $\alpha, \beta \in \mathbb{C}_{\omega_0(\mathbb{T})}$*
$$\frac{\mathbf{G}(\alpha) - \mathbf{G}(\beta)}{\alpha - \beta} = -C(\alpha I - A)^{-1}(\beta I - A)^{-1} B, \tag{2.3}$$
or equivalently $\mathbf{G}'(\alpha) = -C(\alpha I - A)^{-2} B$.
- *$\mathbf{G}$ is bounded on $\mathbb{C}_\omega$ for every $\omega > \omega_0(\mathbb{T})$.*

Note that according to the second statement, $\mathbf{G}$ is determined by A, B and C up to an additive constant.

For any $C \in \mathcal{L}(X_1, Y)$, we define its *$\Lambda$-extension* C_Λ by

$$C_\Lambda z_0 = \lim_{\lambda \to \infty} C\lambda(\lambda I - A)^{-1} z_0.$$

We denote $\mathcal{D}(C_\Lambda)$ its domain, consisting of all $z_0 \in X$ for which the above limit exists. Then we have the following result (see Theorem 3.2 of [33] and [36])

Proposition 2.5 *With the previous notation, if $u \in \mathcal{U}_{\ell oc}$, and $z_0 \in X$, then for almost every $t \geq 0$*

$$y(t) = C_\Lambda \left[z(t) - (\beta I - A)^{-1} Bu(t) \right] + \mathbf{G}(\beta)u(t) \quad \forall\, \beta \in \mathbb{C}_{\omega_0(\mathbb{T})}.$$

Furthermore, if $u \in H^1_{0,\ell oc}([0, \infty), U)$,

$$y(t) = C_\Lambda \mathbb{T}_t z_0 + C \left[\Phi_t u - (\beta I - A)^{-1} Bu(t) \right] + \mathbf{G}(\beta)u(t) \quad \forall\, \beta \in \mathbb{C}_{\omega_0(\mathbb{T})}. \quad (2.4)$$

Curtain and Weiss [12] have given necessary and sufficient conditions for a triple of operators (A, B, C) to be *well posed* (i.e. to be associated with a well posed linear system Σ). We need the definition of admissibility for control and observation operators before stating the theorem.

Definition 2.6 Let X, U and Y be Hilbert spaces. Let A be the generator of a C_0-semigroup $\mathbb{T}$ on X, $B \in \mathcal{L}(U, X_{-1})$ a control operator and $C \in \mathcal{L}(X_1, Y)$ an observation operator.

– B is an *admissible* control operator for $\mathbb{T}$ if and only if for some (and hence any) $\tau > 0$, the operator Φ_τ, defined by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau - s} Bu(s) \mathrm{d}s \quad \forall\, u \in \mathcal{U}_{\ell oc},$$

has its range in X.

– C is an *admissible* observation operator for $\mathbb{T}$ if and only if for some (and hence any) $\tau > 0$, the operator Ψ_τ defined by

$$(\Psi_\tau z_0)(t) = \begin{cases} C\mathbb{T}_t z_0 & \forall\, t \in [0, \tau] \\ 0 & \forall\, t > \tau \end{cases} \quad \forall\, z_0 \in X_1,$$

has a continuous extension to X.

Remark 2.7 It is clear that C is an admissible observation operator for $\mathbb{T}$ if and only if C^* is an admissible control operator for $\mathbb{T}^*$.

Theorem 2.8 (Generating triple, Theorem 5.1 of [11]) *Let X, U and Y be three Hilbert spaces.*

A triple of operators (A, B, C) is well posed (i.e. associated with a well posed linear system Σ) if:

1. A is the generator of a C_0-semigroup $\mathbb{T}$ on X,
2. $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$,
3. $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$,
4. there is an $\alpha \in \mathbb{R}$ such that some (and hence any) solution $\mathbf{G} : \rho(A) \to \mathcal{L}(U, Y)$ of the equation (2.3) is bounded on $\mathbb{C}_\alpha$ (i.e. $\mathbf{G}$ is proper).

Conversely, if Σ is a well posed linear system, with associated triple of operators (A, B, C) and the transfer function $\mathbf{G}$, then the four previous conditions are satisfied.

2.2 Optimizability, estimatability, controllability and observability

It is well-known that for any C_0-semigroup $\mathbb{T}$, we have the following property:

$$\forall \omega > \omega_0(\mathbb{T}), \quad \exists M_\omega \geq 1 : \quad \|\mathbb{T}_t z_0\| \leq M_\omega e^{\omega t} \|z_0\| \quad \forall\, z_0 \in X.$$

If we have $\omega_0(\mathbb{T}) < 0$, then there is $\omega < 0$ satisfying this inequality and the C_0-semigroup will decay exponentially in time. This justifies the following definition.

Definition 2.9 A well posed linear system Σ is *exponentially stable* if and only if $\omega_0(\mathbb{T}) < 0$.

Let us recall some definitions, which can be found in Weiss and Rebarber [44].

Definition 2.10 Let X, U and Y be Hilbert spaces. Let A be the generator of a C_0-semigroup $\mathbb{T}$ on X, $B \in \mathcal{L}(U, X_{-1})$ an admissible control operator for $\mathbb{T}$ and $C \in \mathcal{L}(X_1, Y)$ an admissible observation operator for $\mathbb{T}$.

- The pair (A, B) is *optimizable* if for every $z_0 \in X$, there exists a $u \in \mathcal{U}$ such that $z \in L^2([0, \infty), X)$, where

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-s} B u(s) \mathrm{d}s.$$

- The pair (A, C) is *estimatable* if (A^*, C^*) is optimizable.

A well posed linear system Σ is said to be optimizable if its corresponding pair (A, B) is optimizable, and estimatable when its corresponding pair (A, C) is estimatable.

Definition 2.11 Let X, U and Y be Hilbert spaces. Let A be the generator of a C_0-semigroup $\mathbb{T}$ on X, $B \in \mathcal{L}(U, X_{-1})$ an admissible control operator for $\mathbb{T}$ and $C \in \mathcal{L}(X_1, Y)$ an admissible observation operator for $\mathbb{T}$.

- The pair (A, B) is *exactly controllable* in time $\tau > 0$ if $\operatorname{Ran} \Phi_\tau = X$. It is *approximately controllable* in time $\tau > 0$ if $\overline{\operatorname{Ran} \Phi_\tau} = X$.
- The pair (A, C) is *exactly observable* in time $\tau > 0$ if there exists a constant $k_\tau > 0$ such that

$$\|\Psi_\tau z_0\| \geq k_\tau \|z_0\| \quad \forall\, z_0 \in X.$$

  It is *approximately observable* in time $\tau > 0$ if $\operatorname{Ker} \Psi_\tau = \{0\}$.

Remark 2.12 The pair (A, C) is exactly observable (approximately observable) if and only if (A^*, C^*) is exactly controllable (approximately controllable).

2.3 The dual system

We introduce now the dual system of a well posed linear system.

Theorem 2.13 (Theorem 4 of [45]) *Let* $\Sigma = \begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix}$ *be a well posed linear system with input space* U*, state space* X *and output space* Y*. Define* $\Sigma^d = \left(\Sigma_t^d\right)_{t\geq 0}$ *by*

$$\Sigma_t^d = \begin{bmatrix} \mathbb{T}_t^d & \Phi_t^d \\ \Psi_t^d & \mathbb{F}_t^d \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \text{Я}_t \end{bmatrix} \begin{bmatrix} \mathbb{T}_t^* & \Psi_t^* \\ \Phi_t^* & \mathbb{F}_t^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \text{Я}_t \end{bmatrix}. \tag{2.5}$$

Then, $\Sigma^d = \begin{bmatrix} \mathbb{T}^d & \Phi^d \\ \Psi^d & \mathbb{F}^d \end{bmatrix}$ *is a well posed linear system with input space* Y*, state space* X *and output space* U*. In particular,* $\omega_0(\mathbb{T}) = \omega_0(\mathbb{T}^d)$*. The linear system* Σ^d *is called the dual system of* Σ*.*

Proposition 2.14 (Proposition 4 of [45]) *If* A*,* B *and* C *are respectively the semigroup generator, control operator and observation operator of the well posed linear system* Σ *with growth bound* $\omega_0(\mathbb{T})$*, then the corresponding operators for* Σ^d *are* A^**,* C^* *and* B^**. The transfer functions are related by*

$$\mathbf{G}^d(s) = \mathbf{G}^*(\overline{s}) \quad \forall\, s \in \mathbb{C}_{\omega_0(\mathbb{T})}.$$

2.4 Feedback law

The results of this subsection allow us to construct the forward and backward observers in the framework of well posed linear systems.

Definition 2.15 Let Σ be a well posed linear system with input space U, state space X, output space Y and transfer function $\mathbf{G}$. An operator $K \in \mathcal{L}(Y, U)$ is called an *admissible feedback operator* for Σ if $I - \mathbf{G}K$ has a well posed inverse on some right half-plane (equivalently, if $I - K\mathbf{G}$ has a well posed inverse).

Theorem 2.16 (Theorem 6.1 of [41]) *If* K *is an admissible feedback operator for a well posed linear system* Σ*, the closed-loop system* Σ^K, i.e. Σ with the output feedback $u = Ky + v$ (v is the new control), is well posed. Furthermore, we have

$$\Sigma^K - \Sigma = \Sigma \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma^K = \Sigma^K \begin{bmatrix} 0 & 0 \\ 0 & K \end{bmatrix} \Sigma. \tag{2.6}$$

Under some assumptions, Curtain and Weiss [12, Theorem 5.8] proved that the colocated feedback law exponentially stabilizes the well posed linear system. This generalizes, in some sense, the known results when A is skew-adjoint and C is bounded (see Liu [28]). We give a simpler version of this result, in our particular case.

Theorem 2.17 *Suppose that Σ is a well posed linear system such that A is skew-adjoint, $U = Y$ and $B = C^*$. Then, there exists a $\kappa > 0$ (possibly $\kappa = +\infty$) such that for all $\gamma \in (0, \kappa)$, the feedback law $-\gamma y + v$ (v is the new control) leads to a closed-loop system Σ^γ which is well posed.*

Moreover, if Σ is optimizable and estimatable, then the closed-loop system Σ^γ is exponentially stable.

Remark 2.18 The value of κ is explicitly given in [12, Theorem 5.8]. We have $\kappa = \|E^+\|^{-1}$, where E^+ is the positive part of the self-adjoint operator

$$E = -\frac{1}{2}\left[\mathbf{G}^*(\lambda) + \mathbf{G}(\lambda)\right] + \lambda C\,(\lambda I + A)^{-1}\,(\lambda I - A)^{-1}\,C^* \quad \forall\,\lambda > 0. \tag{2.7}$$

Furthermore, if $0 \in \rho(A)$, then

$$E = -\frac{1}{2}\left[\mathbf{G}^*(0) + \mathbf{G}(0)\right].$$

3 Algorithm of reconstruction

From now on, we suppose that $\Sigma = \begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix}$ is a well posed linear system with input space U, state space X, output space Y, determined by the operators (A, B, C) and the transfer function $\mathbf{G}$, such that

1. A is skew-adjoint,
2. $U = Y$ and $B = C^*$.

Note that from Stone's Theorem, A is the generator of a unitary C_0-group, which will be denoted by $\mathbb{S}$. In the sequel, we suppose without loss of generality that the control u of Σ satisfies $u \equiv 0$.

3.1 The forward and backward observers

Let us begin with a forward observer Σ^+ of Σ (corresponding to (1.4)). With the above assumptions, we apply Theorem 2.17 to define the closed-loop system Σ^+ for some $\gamma \in (0, \kappa)$.

In the first section of this paper, we have seen that the forward error $e^+(t) = z^+(t) - z(t)$ satisfies $\dot{e}^+ = (A - \gamma C^* C)\, e^+$ by simple algebraic computations. Here, $A - \gamma C^* C$ has no more meaning, since C is unbounded. Therefore, we use directly the definitions of the trajectories z and z^+ to show that $e^+(t) = \mathbb{T}_t^+ e(0)$.

We denote by $\begin{bmatrix} \mathbb{T}^+ & \Phi^+ \\ \Psi^+ & \mathbb{F}^+ \end{bmatrix}$ the operators of Σ^+. Then from (2.6) with $K = -\gamma I$, we have

$$\mathbb{T}_t^+ z_0^+ = \mathbb{S}_t z_0^+ - \gamma \Phi_t \Psi_t^+ z_0^+ = \mathbb{S}_t z_0^+ - \gamma \Phi_t^+ \Psi_t z_0^+ \quad \forall\, z_0^+ \in X. \tag{3.1}$$

Let us denote by z and z_0, respectively by z^+ and z_0^+, the trajectory and initial state of Σ, respectively Σ^+. We add the control $v = \gamma y$ to Σ^+, where y is the output function of the initial system Σ. Note that $y = \Psi_\infty z_0$ since we suppose that $u \equiv 0$ (see (2.1) in Definition 2.2). We have

$$z(t) = \mathbb{S}_t z_0, \qquad z^+(t) = \mathbb{T}_t^+ z_0^+ + \gamma \Phi_t^+ y \quad \forall\, z_0, z_0^+ \in X.$$

From the above equalities and $\Phi_t^+ y = \Phi_t^+ \mathbf{P}_t y = \Phi_t^+ \mathbf{P}_t \Psi_\infty z_0 = \Phi_t^+ \Psi_t z_0$, we can rewrite

$$z^+(t) = \mathbb{S}_t z_0^+ - \gamma \Phi_t^+ \Psi_t \left(z_0^+ - z_0\right) \quad \forall\, z_0, z_0^+ \in X.$$

Then, we call Σ^+ a *forward observer* of Σ, since under some additional assumptions, $z^+(t) \to z(t)$ as $t \to \infty$. Indeed, $e^+(t) = z^+(t) - z(t)$ satisfies

$$e^+(t) = \mathbb{S}_t \left(z_0^+ - z_0\right) - \gamma \Phi_t^+ \Psi_t \left(z_0^+ - z_0\right) = \mathbb{T}_t^+ \left(z_0^+ - z_0\right) \quad \forall\, z_0, z_0^+ \in X,$$

and following Theorem 2.17, $\mathbb{T}^+$ is exponentially stable if (and only if) Σ is optimizable and estimatable.

Now, the idea is to go back in time, starting from $z_\tau^- = \mathbb{T}_\tau^+ z_0^+$ for a fixed finite time $\tau > 0$. Thus, we have to define a backward observer Σ^- of Σ (corresponding to (1.5)). We first define $\Sigma^d = \begin{bmatrix} \mathbb{T}^d & \Phi^d \\ \Psi^d & \mathbb{F}^d \end{bmatrix}$, the dual system of Σ, using Theorem 2.13. From Proposition 2.14, the C_0-semigroup generator of Σ^d is $A^* = -A$, and then the C_0-semigroup of Σ^d is $\mathbb{S}^{-1} = (\mathbb{S}_{-t})_{t \geq 0}$. From our assumptions, the control and observation operators of Σ^d are the same as those of Σ.

Before the definition of Σ^-, we give the following lemma, immediate from (2.7), which shows that the same parameter γ can be used for both Σ^+ and Σ^-.

Lemma 3.1 *Let Σ be a well posed linear system verifying the assumptions of the beginning of this section, and Σ^d its dual system. Denote κ and κ^d the maximal bound for γ in Theorem 2.17, for Σ and Σ^d, respectively. Then $\kappa = \kappa^d$.*

From now on, we take the same parameter $\gamma \in (0, \kappa)$ for both Σ^+ and Σ^-.

We define by Σ^- the closed-loop system of Σ^d, for some $\gamma \in (0, \kappa)$. We denote by $\begin{bmatrix} \mathbb{T}^- & \Phi^- \\ \Psi^- & \mathbb{F}^- \end{bmatrix}$ the operators of Σ^-. Then from (2.6) with $K = -\gamma I$, we have

$$\mathbb{T}_t^- z_\tau^- = \mathbb{S}_{-t} z_\tau^- - \gamma \Phi_t^d \Psi_t^- z_\tau^- = \mathbb{S}_{-t} z_\tau^- - \gamma \Phi_t^- \Psi_t^d z_\tau^- \quad \forall\, z_\tau^- \in X. \tag{3.2}$$

Denote by z^- the trajectory of Σ^- with the control $v = \gamma\, Я_\tau\, y$. We know that $\Phi_\tau^-\, Я_\tau\, y = \Phi_\tau^-\, Я_\tau\, \Psi_\tau z_0$ and it is easy to see that $Я_\tau\, \Psi_\tau = \Psi_\tau^d \mathbb{S}_\tau$, for all $\tau \geq 0$. Then, we get

$$z^-(\tau) = \mathbb{S}_{-\tau} z_\tau^- - \gamma \Phi_\tau^- \Psi_\tau^d \left(z_\tau^- - \mathbb{S}_\tau z_0\right).$$

Setting $e^-(t) = \left(\mathcal{R}_\tau z^-\right)(t) - z(t)$, we obtain

$$\begin{aligned} e^-(0) &= z^-(\tau) - z_0 \\ &= \mathbb{S}_{-\tau} z_\tau^- - \gamma \Phi_\tau^- \Psi_\tau^d \left(z_\tau^- - \mathbb{S}_\tau z_0\right) - \mathbb{S}_{-\tau}\mathbb{S}_\tau z_0 \\ &= \mathbb{S}_{-\tau}\left(z_\tau^- - \mathbb{S}_\tau z_0\right) - \gamma \Phi_\tau^- \Psi_\tau^d \left(z_\tau^- - \mathbb{S}_\tau z_0\right) \\ &= \mathbb{T}_\tau^- \left(z_\tau^- - z(\tau)\right). \end{aligned}$$

And since $z_\tau^- = z^+(\tau) = \mathbb{T}_\tau^+ z_0^+$, we finally obtain

$$e^-(0) = \mathbb{T}_\tau^- \mathbb{T}_\tau^+ \left(z_0^+ - z_0\right).$$

If Σ is optimizable and estimatable, then there exists a $\tau > 0$ such that $\|\mathbb{T}_\tau^- \mathbb{T}_\tau^+\|_{\mathcal{L}(X)} < 1$ (since $\mathbb{T}^+$ and $\mathbb{T}^-$ are then exponentially stable). In other words, $z^-(0)$ is a better approximation of z_0 than z_0^+. The iteration of this process gives a method to reconstruct z_0 with exponential decay of the error, as after n iterations we have

$$\|e_n^-(0)\| \le \left\|\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right\|_{\mathcal{L}(X)}^n \left\|z_0^+ - z_0\right\| \quad \forall\, n \in \mathbb{N}.$$

3.2 Relation between Σ^+ and Σ^-

In this subsection we prove the following theorem, which will be useful in many computations.

Theorem 3.2 *With the assumptions given at the beginning of this section, we have* $\left(\Sigma^+\right)^d = \Sigma^-$.

The proof of this result is based on the following equalities.

Lemma 3.3 *With the assumptions and notation of Theorem 3.2, we have*

$$\left(I + \gamma \mathbb{F}_\tau\right)^{-1} \Psi_\tau = \mathcal{R}_\tau \left(I + \gamma \mathcal{R}_\tau \mathbb{F}_\tau \mathcal{R}_\tau\right)^{-1} \mathcal{R}_\tau \Psi_\tau, \tag{3.3}$$

$$\mathbb{F}_\tau \left(I + \gamma \mathbb{F}_\tau\right)^{-1} = \mathbb{F}_\tau \mathcal{R}_\tau \left(I + \gamma \mathcal{R}_\tau \mathbb{F}_\tau \mathcal{R}_\tau\right)^{-1} \mathcal{R}_\tau. \tag{3.4}$$

Proof Remark that from (2.6),

$$\left(I + \gamma \mathbb{F}_\tau\right)\left(I - \gamma \mathbb{F}_\tau^+\right) = I = \left(I - \gamma \mathbb{F}_\tau^+\right)\left(I + \gamma \mathbb{F}_\tau\right),$$

showing that $\left(I + \gamma \mathbb{F}_\tau\right)^{-1} = I - \gamma \mathbb{F}_\tau^+$.

On the other hand, we easily obtain that

$$\begin{aligned} \left(I - \gamma \mathcal{R}_\tau \mathbb{F}_\tau^+ \mathcal{R}_\tau\right)\left(I + \gamma \mathcal{R}_\tau \mathbb{F}_\tau \mathcal{R}_\tau\right) &= I + \gamma \mathcal{R}_\tau \underbrace{\left(\mathbb{F}_\tau - \mathbb{F}_\tau^+ - \gamma \mathbb{F}_\tau^+ \mathbb{F}_\tau\right)}_{=0 \quad \text{from (2.6)}} \mathcal{R}_\tau \\ &= \left(I + \gamma \mathcal{R}_\tau \mathbb{F}_\tau \mathcal{R}_\tau\right)\left(I - \gamma \mathcal{R}_\tau \mathbb{F}_\tau^+ \mathcal{R}_\tau\right). \end{aligned}$$

In other words, $(I + \gamma \mathcal{R}_\tau \mathbb{F}_\tau \mathcal{R}_\tau)^{-1} = I - \gamma \mathcal{R}_\tau \mathbb{F}_\tau^+ \mathcal{R}_\tau$; hence, we have to prove that equality (3.3) reduces to

$$\left(I - \gamma \mathbb{F}_\tau^+\right) \Psi_\tau = \mathcal{R}_\tau \left(I - \gamma \mathcal{R}_\tau \mathbb{F}_\tau^+ \mathcal{R}_\tau\right) \mathcal{R}_\tau \Psi_\tau.$$

But

$$\begin{aligned}\mathcal{R}_\tau \left(I - \gamma \mathcal{R}_\tau \mathbb{F}_\tau^+ \mathcal{R}_\tau\right) \mathcal{R}_\tau \Psi_\tau &= \mathbf{P}_\tau \Psi_\tau - \gamma \mathbf{P}_\tau \mathbb{F}_\tau^+ \mathbf{P}_\tau \Psi_\tau \\ &= \Psi_\tau - \gamma \mathbb{F}_\tau^+ \Psi_\tau = \left(I - \gamma \mathbb{F}_\tau^+\right) \Psi_\tau.\end{aligned}$$

Similarly, equality (3.4) reduces to

$$\mathbb{F}_\tau \left(I - \gamma \mathbb{F}_\tau^+\right) = \mathbb{F}_\tau \mathcal{R}_\tau \left(I - \gamma \mathcal{R}_\tau \mathbb{F}_\tau^+ \mathcal{R}_\tau\right) \mathcal{R}_\tau,$$

and

$$\begin{aligned}\mathbb{F}_\tau \mathcal{R}_\tau \left(I - \gamma \mathcal{R}_\tau \mathbb{F}_\tau^+ \mathcal{R}_\tau\right) \mathcal{R}_\tau &= \mathbb{F}_\tau \mathbf{P}_\tau - \gamma \mathbb{F}_\tau \mathbf{P}_\tau \mathbb{F}_\tau^+ \mathbf{P}_\tau \\ &= \mathbb{F}_\tau - \gamma \mathbb{F}_\tau \mathbb{F}_\tau^+ = \mathbb{F}_\tau \left(I - \gamma \mathbb{F}_\tau^+\right).\end{aligned}$$

□

Proof (*Proof of Theorem 3.2*) We have to show that

$$\left(\mathbb{T}_\tau^+\right)^d = \mathbb{T}_\tau^-, \qquad \left(\Phi_\tau^+\right)^d = \Phi_\tau^-, \qquad \left(\Psi_\tau^+\right)^d = \Psi_\tau^-, \qquad \left(\mathbb{F}_\tau^+\right)^d = \mathbb{F}_\tau^-.$$

Let us begin with $\left(\Phi_\tau^+\right)^d = \Phi_\tau^-$. Using $\mathcal{R}_\tau^* = \mathcal{R}_\tau$, (2.5), (2.6) and (3.3), we have

$$\begin{aligned}\left(\Phi_\tau^+\right)^d &= \left(\Psi_\tau^+\right)^* \mathcal{R}_\tau \\ &= \Psi_\tau^* \mathcal{R}_\tau \left(I + \gamma \mathcal{R}_\tau \mathbb{F}_\tau^* \mathcal{R}_\tau\right)^{-1} \mathcal{R}_\tau \mathcal{R}_\tau \\ &= \Phi_\tau^d \left(I + \gamma \mathbb{F}_\tau^d\right)^{-1} \mathbf{P}_\tau \\ &= \Phi_\tau^-.\end{aligned}$$

Similarly, we obtain $\left(\Psi_\tau^+\right)^d = \Psi_\tau^-$. Then, using (3.2), we have $\left(\mathbb{T}_\tau^+\right)^d = \left(\mathbb{T}_\tau^+\right)^* = \mathbb{S}_{-\tau} - \gamma \Phi_\tau^- \Psi_\tau^d = \mathbb{T}_\tau^-$.

It remains to show that $\left(\mathbb{F}_\tau^+\right)^d = \mathbb{F}_\tau^-$. Again, from $\mathcal{R}_\tau^* = \mathcal{R}_\tau$, (2.5), (2.6) and (3.4), we have

$$\begin{aligned}\left(\mathbb{F}_\tau^+\right)^d &= \mathcal{R}_\tau \left(\mathbb{F}_\tau^+\right)^* \mathcal{R}_\tau \\ &= \mathcal{R}_\tau \mathcal{R}_\tau \left(I + \gamma \mathcal{R}_\tau \mathbb{F}_\tau^* \mathcal{R}_\tau\right)^{-1} \mathcal{R}_\tau \mathbb{F}_\tau^* \mathcal{R}_\tau \\ &= \mathbb{F}_\tau^-.\end{aligned}$$

□

3.3 The forward–backward operator

We now study in the general case the *forward–backward operator* $\mathbb{T}_\tau^-\mathbb{T}_\tau^+$, for a fixed τ, with $\gamma \in (0, \kappa)$. In other words, we suppose neither that Σ is optimizable and estimatable, nor that τ is large enough to ensure that $\|\mathbb{T}_\tau^-\mathbb{T}_\tau^+\|_{\mathcal{L}(X)} < 1$.

Let us introduce the following orthogonal decomposition of an element z of X.

Lemma 3.4 *With the previous notation and definitions, we have*

$$X = \operatorname{Ker} \Psi_\tau \oplus \overline{\operatorname{Ran} \Phi_\tau^d}.$$

Proof This follows immediately from the decomposition $X = \operatorname{Ker} \Psi_\tau \oplus (\operatorname{Ker} \Psi_\tau)^\perp = \operatorname{Ker} \Psi_\tau \oplus \overline{\operatorname{Ran} \Psi_\tau^*}$ and from $\Phi_\tau^d = \Psi_\tau^* \,Я_\tau$ (see Eq. (2.5)), since, obviously, $\operatorname{Ran} \Psi_\tau^* = \operatorname{Ran} \left[\Psi_\tau^* Я_\tau\right]$. □

In the sequel of the paper, we denote by $V_{\text{Obs}} = \overline{\operatorname{Ran} \Phi_\tau^d}$ and $V_{\text{Unobs}} = \operatorname{Ker} \Psi_\tau$, which correspond respectively to the observable part and to the unobservable part of an element of X.

Proposition 3.5 *We have*

$$\left(\mathbb{T}_\tau^-\mathbb{T}_\tau^+\right) V_{\text{Obs}} \subset V_{\text{Obs}}, \quad \left(\mathbb{T}_\tau^-\mathbb{T}_\tau^+\right) V_{\text{Unobs}} \subset V_{\text{Unobs}}.$$

Proof From (3.1) and (3.2), we have

$$\mathbb{T}_\tau^-\mathbb{T}_\tau^+ = I - \gamma \mathbb{S}_{-\tau} \Phi_\tau \Psi_\tau^+ - \gamma \Phi_\tau^- \Psi_\tau^d \mathbb{S}_\tau + \gamma^2 \Phi_\tau^- \Psi_\tau^d \Phi_\tau \Psi_\tau^+. \tag{3.5}$$

First, note that from (2.5)

$$\mathbb{S}_{-\tau} \Phi_\tau = \Phi_\tau^d Я_\tau = \Psi_\tau^* \mathbf{P}_\tau.$$

Second, simple computations give $\Psi_\tau^d \mathbb{S}_\tau = Я_\tau \Psi_\tau$ and Theorem 3.2 shows that $\Phi_\tau^- = \left(\Psi_\tau^+\right)^*$. Finally, from (2.6), we see that

$$\Psi_\tau^+ = (I + \gamma \mathbb{F}_\tau)^{-1} \Psi_\tau,$$

and then (3.5) becomes

$$\begin{aligned} \mathbb{T}_\tau^-\mathbb{T}_\tau^+ = I - \gamma \Psi_\tau^* (I + \gamma \mathbb{F}_\tau)^{-1} \Psi_\tau - \gamma \Psi_\tau^* \left(I + \gamma \mathbb{F}_\tau^*\right)^{-1} Я_\tau \Psi_\tau \\ + \gamma^2 \Psi_\tau^* \left(I + \gamma \mathbb{F}_\tau^*\right)^{-1} Я_\tau \Psi_\tau^d \Phi_\tau (I + \gamma \mathbb{F}_\tau)^{-1} \Psi_\tau. \end{aligned}$$

Thus, by Lemma 3.4

$$\left\langle \mathbb{T}_\tau^-\mathbb{T}_\tau^+ z, \theta \right\rangle = \langle z, \theta \rangle = 0, \quad \left\langle \mathbb{T}_\tau^-\mathbb{T}_\tau^+ \theta, z \right\rangle = \langle \theta, z \rangle = 0 \quad \forall\, z \in V_{\text{Obs}}, \quad \theta \in V_{\text{Unobs}},$$

and then $\left(\mathbb{T}_\tau^-\mathbb{T}_\tau^+\right) V_{\text{Obs}} \subset V_{\text{Obs}}$ and $\left(\mathbb{T}_\tau^-\mathbb{T}_\tau^+\right) V_{\text{Unobs}} \subset V_{\text{Unobs}}$. □

Remark 3.6 We point out the fact that $\|\mathbb{T}_\tau^- \mathbb{T}_\tau^+ z\| = \|z\|$ for all $z \in V_{\text{Unobs}}$.

We immediately obtain the following result

Corollary 3.7 *Let Π be the orthogonal projector from X onto V_{Obs}, then*

$$\mathbb{T}_\tau^- \mathbb{T}_\tau^+ \Pi = \Pi \mathbb{T}_\tau^- \mathbb{T}_\tau^+ .$$

Proposition 3.8 *Denote by $L = \left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)|_{V_{\text{Obs}}} \in \mathcal{L}(V_{\text{Obs}})$. Then L is a positive self-adjoint operator on V_{Obs}.*

Proof From Theorem 3.2, we have for all $z_1, z_2 \in X$

$$\left\langle \mathbb{T}_\tau^- \mathbb{T}_\tau^+ z_1, z_2 \right\rangle = \left\langle \mathbb{T}_\tau^+ z_1, \mathbb{T}_\tau^+ z_2 \right\rangle .$$

Then

$$\left\langle \mathbb{T}_\tau^- \mathbb{T}_\tau^+ z, z \right\rangle = \left\| \mathbb{T}_\tau^+ z \right\|^2 \quad \forall\, z \in X. \tag{3.6}$$

Thus $\mathbb{T}_\tau^- \mathbb{T}_\tau^+$ is positive self-adjoint on X, and *a fortiori* L is positive self-adjoint on V_{Obs} (by Proposition 3.5). □

Proposition 3.9 *Let L be as in Proposition 3.8. Then the following statements hold:*

1. *For all $z \in V_{\text{Obs}} \backslash \{0\}$, we have $\|Lz\| < \|z\|$.*
2. *We have the following characterization*

$$\|L\|_{\mathcal{L}(V_{\text{Obs}})} < 1 \quad \Longleftrightarrow \quad V_{\text{Obs}} = \overline{\text{Ran}\, \Phi_\tau^d} = \text{Ran}\, \Phi_\tau^d .$$

We need two lemmas to prove this proposition.

Lemma 3.10 *Let $\Sigma = \begin{bmatrix} \mathbb{T} & \Phi \\ \Psi & \mathbb{F} \end{bmatrix}$ be a well posed linear system satisfying the assumptions of the beginning of this section. We have for all $u \in \mathcal{U}_{\ell oc}$*

$$\left(\Phi_\tau^* \Phi_\tau - \mathbb{F}_\tau^* - \mathbb{F}_\tau\right) u(t) = 2Eu(t) \quad \textit{for a.e.}\ \ t \in (0, \tau),$$

where E is the self-adjoint operator defined by (2.7).

Proof Let $\Sigma^d = \begin{bmatrix} \mathbb{T}^d & \Phi^d \\ \Psi^d & \mathbb{F}^d \end{bmatrix}$ be the dual system of Σ. We first remark that

$$\Phi_\tau^* \Phi_\tau - \mathbb{F}_\tau^* = \text{Я}_\tau \Psi_\tau^d \Phi_\tau - \text{Я}_\tau \mathbb{F}_\tau^d \text{Я}_\tau .$$

Let u be a control belonging to

$$\mathcal{H}_\tau = \{w \in H^1_{\ell oc}([0, \infty), Y) \mid w(0) = w(\tau) = 0\},$$

and z the trajectory of Σ with null initial state and control u. Then, z satisfies

$$\begin{cases} \dot{z}(t) = Az(t) + C^* u(t) & \forall\, t \in [0, \tau], \\ z(0) = 0, \end{cases}$$

and the output of Σ is given by $y = y|_{[0,\tau]}(t) = (\mathbb{F}_\tau u)\,(t)$.

Now, we consider $z^d(t) = \mathcal{R}_\tau\, z(t) = z(\tau - t)$. Then, $z^d(t)$ is the trajectory of Σ^d with control $v = -\,\mathcal{R}_\tau\, u$ and initial state $\Phi_\tau u$

$$\begin{cases} \dot{z}^d(t) = -Az^d(t) - C^*\mathcal{R}_\tau u(t) & \forall\, t \in [0, \tau], \\ z^d(0) = \Phi_\tau u. \end{cases}$$

The output of Σ^d is then $y^d = \Psi_\tau^d \Phi_\tau u - \mathbb{F}_\tau^d\, \mathcal{R}_\tau\, u$.

Now, we have that

$$\mathcal{R}_\tau y^d - y = \mathcal{R}_\tau \Psi_\tau^d \Phi_\tau u - \mathcal{R}_\tau \mathbb{F}_\tau^d \mathcal{R}_\tau u - \mathbb{F}_\tau u.$$

Since $u \in \mathcal{H}_\tau$, we have in particular that u and $\mathcal{R}_\tau\, u$ belong to $H^1_{0,\ell oc}([0,\infty), Y)$ and from (2.4), with $\beta = \lambda > 0$, we have for almost every $t \in (0, \tau)$

$$\begin{aligned} \mathcal{R}_\tau y^d(t) - y(t) = \mathcal{R}_\tau C_\Lambda^d \mathbb{S}_{-t} \Phi_\tau u + \mathcal{R}_\tau C \left[\Phi_t^d u + (\lambda I + A)^{-1} C^* \mathcal{R}_\tau u(t)\right] \\ -C\left[\Phi_t u - (\lambda I - A)^{-1} C^* u(t)\right] - (\mathbf{G}^*(\lambda) + \mathbf{G}(\lambda)) u(t). \end{aligned}$$

But C_Λ^d is an extension of C, thus we can rewrite the above equality

$$\begin{aligned} \mathcal{R}_\tau y^d(t) - y(t) = C_\Lambda^d\big[\mathcal{R}_\tau z^d(t) + (\lambda I + A)^{-1} C^* \mathcal{R}_\tau^2 u(t) \\ - z(t) + (\lambda I - A)^{-1} C^* u(t)\big] - (\mathbf{G}^*(\lambda) + \mathbf{G}(\lambda)) u(t) \quad \text{for a.e. } t \in (0, \tau), \end{aligned}$$

Since $z^d(t) = \mathcal{R}_\tau\, z(t)$ and $\mathcal{R}_\tau^2\, u(t) = u(t)$ on $(0, \tau)$, this becomes

$$\begin{aligned} \mathcal{R}_\tau y^d(t) - y(t) = C_\Lambda^d\big[\,(\lambda I + A)^{-1} + (\lambda I - A)^{-1}\,\big] C^* u(t) \\ -(\mathbf{G}^*(\lambda) + \mathbf{G}(\lambda)) u(t) \quad \text{for a.e. } \ t \in (0, \tau), \end{aligned}$$

Now, using $(\lambda I - A)^{-1} = -(-\lambda I + A)^{-1}$ for all $\lambda > 0$ and the resolvent identity, we get

$$\begin{aligned} \mathcal{R}_\tau y^d(t) - y(t) = 2\lambda C_\Lambda^d\, (\lambda I + A)^{-1} (\lambda I - A)^{-1} C^* u(t) \\ -(\mathbf{G}^*(\lambda) + \mathbf{G}(\lambda)) u(t) \quad \text{for a.e. } \ t \in (0, \tau), \end{aligned}$$

But $(\lambda I + A)^{-1} (\lambda I - A)^{-1} C^* \in \mathcal{L}(Y, X_1)$, and thus we can replace C_Λ^d by C in the above equality, and (2.7) gives the result

$$\mathcal{R}_\tau y^d(t) - y(t) = 2Eu(t) \quad \text{for a.e. } \ t \in (0, \tau),$$

We conclude by the density of $\mathcal{H}_\tau$ in $\mathcal{U}_{\ell oc}$. □

Finally, we recall how to characterize the closure of the range of a bounded linear operator. We give this lemma without proof (see for instance Brézis [8, Chapter 2]).

Lemma 3.11 *A bounded linear operator $T \in \mathcal{L}(Z_1, Z_2)$, where Z_1 and Z_2 are Hilbert spaces, has a closed range if and only if there exists a constant $k > 0$ such that*

$$\|T^* f\| \geq k\|(I - P)f\| \quad \forall\, f \in Z_2, \tag{3.7}$$

where P is the orthogonal projector on Ker T^*.

We are now able to prove Proposition 3.9.

Proof (*Proof of Proposition 3.9*) The two points of Proposition 3.9 are consequences of the following relation

$$\|\mathbb{T}_\tau^+ z\|^2 = \|z\|^2 - 2\gamma\|\Psi_\tau^+ z\|^2 + 2\gamma^2\left\langle E\Psi_\tau^+ z, \Psi_\tau^+ z\right\rangle \quad \forall\, z \in X, \tag{3.8}$$

where E is the self-adjoint operator of Remark 2.18.

Let us begin with the proof of (3.8).

$$\begin{aligned}\|\mathbb{T}_\tau^+ z\|^2 &= \|\mathbb{S}_\tau z - \gamma\Phi_\tau\Psi_\tau^+ z\|^2\\ &= \|\mathbb{S}_\tau z\|^2 - \gamma\langle\Phi_\tau^*\mathbb{S}_\tau z, \Psi_\tau^+ z\rangle - \gamma\langle\Psi_\tau^+ z, \Phi_\tau^*\mathbb{S}_\tau z\rangle + \gamma^2\|\Phi_\tau\Psi_\tau^+ z\|^2.\end{aligned}$$

From $\|\mathbb{S}_\tau z\| = \|z\|$, $\Phi_\tau^*\mathbb{S}_\tau = \Psi_\tau$ and $\Psi_\tau = (I + \gamma\mathbb{F}_\tau)\,\Psi_\tau^+$, we obtain

$$\begin{aligned}\|\mathbb{T}_\tau^+ z\|^2 &= \|z\|^2 - \gamma\langle(I+\gamma\mathbb{F}_\tau)\,\Psi_\tau^+ z, \Psi_\tau^+ z\rangle - \gamma\langle\Psi_\tau^+ z, (I+\gamma\mathbb{F}_\tau)\,\Psi_\tau^+ z\rangle + \gamma^2\|\Phi_\tau\Psi_\tau^+ z\|^2\\ &= \|z\|^2 - 2\gamma\|\Psi_\tau^+ z\|^2 + \gamma^2\left\langle\left(\Phi_\tau^*\Phi_\tau - \mathbb{F}_\tau - \mathbb{F}_\tau^*\right)\Psi_\tau^+ z, \Psi_\tau^+ z\right\rangle.\end{aligned}$$

We use now Lemma 3.10 to get (3.8).

We denote E^+ the positive part of E, then

$$\begin{aligned}\|\mathbb{T}_\tau^+ z\|^2 &= \|z\|^2 - 2\gamma\|\Psi_\tau^+ z\|^2 + 2\gamma^2\left\langle E\Psi_\tau^+ z, \Psi_\tau^+ z\right\rangle\\ &\leq \|z\|^2 - 2\gamma\|\Psi_\tau^+ z\|^2 + 2\gamma^2\underbrace{\|E^+\|}_{=\kappa^{-1}}\|\Psi_\tau^+ z\|^2\\ &\leq \|z\|^2 - 2\gamma\left(1 - \gamma\kappa^{-1}\right)\|\Psi_\tau^+ z\|^2.\end{aligned}$$

where κ is the maximum bound for γ, given in Remark 2.18. In particular we have $1 - \gamma\kappa^{-1} > 0$. From this, if $z \in V_{\text{Obs}}\backslash\{0\}$, thus $\|\Psi_\tau^+ z\|^2 > 0$, and therefore

$$\begin{aligned}\|Lz\|^2 &= \langle Lz, Lz\rangle\\ &= \left\langle L\,(L)^{\frac{1}{2}}\,z, (L)^{\frac{1}{2}}\,z\right\rangle\\ &= \|\mathbb{T}_\tau^+\,(L)^{\frac{1}{2}}\,z\|^2\\ &\leq \|\,(L)^{\frac{1}{2}}\,z\|^2 - 2\gamma\left(1 - \gamma\kappa^{-1}\right)\|\Psi_\tau^+\,(L)^{\frac{1}{2}}\,z\|^2\\ &\leq \langle Lz, z\rangle\\ &\leq \|\mathbb{T}_\tau^+ z\|^2\\ &\leq \|z\|^2 - 2\gamma\left(1 - \gamma\kappa^{-1}\right)\|\Psi_\tau^+ z\|^2\\ &< \|z\|^2.\end{aligned}$$

Thus, the first point is shown.

For the second point, we use the fact (from $\Psi_\tau = (I + \gamma \mathbb{F}_\tau)\,\Psi_\tau^+$ and $\Psi_\tau^+ = \left(I - \gamma \mathbb{F}_\tau^+\right)\Psi_\tau$) that there exist two constants $m, M > 0$ such that

$$m\|\Psi_\tau z\| \leq \|\Psi_\tau^+ z\| \leq M\|\Psi_\tau z\| \forall\, z \in X,$$

together with Lemma 3.11 to get that

$$\operatorname{Ran} \Phi_\tau^d = V_{\mathrm{Obs}} \iff \inf_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \|\Psi_\tau z\| > 0 \iff \inf_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \|\Psi_\tau^+ z\| > 0. \tag{3.9}$$

Since L is self-adjoint and positive, we have

$$\begin{aligned} \|L\|_{\mathcal{L}(V_{\mathrm{Obs}})} &= \sup_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \langle Lz, z\rangle \\ &= \sup_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \|\mathbb{T}_\tau^+ z\|^2 \\ &\leq 1 - 2\gamma\left(1 - \gamma\kappa^{-1}\right) \inf_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \|\Psi_\tau^+ z\|. \end{aligned}$$

So that from (3.9)

$$\operatorname{Ran} \Phi_\tau^d = V_{\mathrm{Obs}} \implies \|L\|_{\mathcal{L}(V_{\mathrm{Obs}})} < 1.$$

Conversely, from (3.8), we get

$$\|\mathbb{T}_\tau^+ z\|^2 = \|z\|^2 - 2\gamma\left\langle (I - \gamma E)\,\Psi_\tau^+ z, \Psi_\tau^+ z\right\rangle, \tag{3.10}$$

and since

$$\|\mathbb{T}_\tau^+ z\|^2 < \|z\|^2 \quad \forall\, z \in V_{\mathrm{Obs}}\backslash\{0\},$$

we see that

$$\left\langle (I - \gamma E)\,\Psi_\tau^+ z, \Psi_\tau^+ z\right\rangle > 0 \quad \forall\, z \in V_{\mathrm{Obs}}\backslash\{0\}. \tag{3.11}$$

Thus,

$$0 < \left\langle (I - \gamma E)\,\Psi_\tau^+ z, \Psi_\tau^+ z\right\rangle \leq \|I - \gamma E\|\,\|\Psi_\tau^+ z\|^2,$$

which shows that

$$\inf_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \|\Psi_\tau^+ z\| = 0 \implies \inf_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \left\langle (I - \gamma E)\,\Psi_\tau^+ z, \Psi_\tau^+ z\right\rangle = 0$$

and then, if $\operatorname{Ran} \Phi_\tau^d$ is not closed in X, from (3.9) and the above relation

$$\begin{aligned} \|L\|_{\mathcal{L}(V_{\mathrm{Obs}})} &= \sup_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \|\mathbb{T}_\tau^+ z\|^2 \\ &= 1 - 2\gamma \inf_{z\in V_{\mathrm{Obs}},\ \|z\|=1} \left\langle (I - \gamma E)\,\Psi_\tau^+ z, \Psi_\tau^+ z\right\rangle \\ &= 1. \end{aligned}$$

So that

$$\text{Ran}\,\Phi_\tau^d \neq V_{\text{Obs}} \implies \|L\|_{\mathcal{L}(V_{\text{Obs}})} = 1,$$

or in other words

$$\|L\|_{\mathcal{L}(V_{\text{Obs}})} < 1 \implies \text{Ran}\,\Phi_\tau^d = V_{\text{Obs}},$$

and Proposition 3.9 is proved. □

3.4 Proofs of the main results

Proof (*Proof of Theorem 1.1*) Let $z_0, z_0^+ \in X$. From Lemma 3.4, we can write uniquely $z_0 = \Pi z_0 + (I - \Pi)\, z_0$ and $z_0^+ = \Pi z_0^+ + (I - \Pi)\, z_0^+$. We will successively prove assertions 1., 3. and 2., in this order.

With the notation of Propositions 3.5, 3.8 shows that the error (1.6) can be rewritten for all $n \in \mathbb{N}$ as

$$\left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)^n \left(z_0^+ - z_0\right) = L^n \Pi \left(z_0^+ - z_0\right) + \left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)^n (I - \Pi) \left(z_0^+ - z_0\right). \qquad (3.12)$$

1. First, we prove that the first term of the right-hand side of (3.12) has no contribution. From Remark 3.6, we know that

$$\|\mathbb{T}_\tau^- \mathbb{T}_\tau^+ z\| = \|z\| \quad \forall\, z \in V_{\text{Unobs}}.$$

Using Proposition 3.5, we iterate and obtain

$$\left\|\left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)^n z\right\| = \|z\| \quad \forall\, n \in \mathbb{N}, z \in V_{\text{Unobs}}.$$

Finally, from Corollary 3.7, we get

$$\begin{aligned}\left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)^n (I - \Pi) \left(z_0^+ - z_0\right) &= (I - \Pi) \left(\mathbb{T}_\tau^- \mathbb{T}_\tau^+\right)^n \left(z_0^+ - z_0\right)\\ &= (I - \Pi) \left(z_n^-(0) - z_0\right),\end{aligned}$$

and the first part of the theorem is proved.

3. Let $z \in V_{\text{Obs}} = \overline{\text{Ran}\,\Phi_\tau^d}$. From the second statement in Proposition 3.9, $\overline{\text{Ran}\,\Phi_\tau^d} = \text{Ran}\,\Phi_\tau^d$ if and only if $\|L\|_{\mathcal{L}(V_{\text{Obs}})} < 1$. Then, if $\text{Ran}\,\Phi_\tau^d$ is closed in X, we have for all $n \in \mathbb{N}$

$$\left\|L^n z\right\| \leq \alpha^n \|z\| \quad \forall\, z \in V_{\text{Obs}},$$

with $\alpha = \|L\|_{\mathcal{L}(V_{\text{Obs}})} < 1$. Conversely, if the above relation holds for all $n \in \mathbb{N}$, then $\|L\|_{\mathcal{L}(V_{\text{Obs}})} \leq \alpha < 1$ (taking $n = 1$), and the last statement in Proposition 3.9 shows that $\text{Ran}\,\Phi_\tau^d$ is closed in X. The last part of the theorem is then proved.

2. We suppose now that Ran Φ_τ^d is not closed in X. We know from Proposition 3.8 that L is self-adjoint, positive, and so is L^n for all $n \in \mathbb{N}$. In particular, for all $n \in \mathbb{N}$, we have $\|L^n\|_{\mathcal{L}(V_{\mathrm{Obs}})} = \|L\|^n_{\mathcal{L}(V_{\mathrm{Obs}})} = 1$. Iterating $n \in \mathbb{N}$ times (3.10), we obtain

$$\left\langle L^n z, z\right\rangle = \|z\|^2 - 2\gamma \sum_{k=1}^{n} \left\langle (I-\gamma E)\, \Psi_\tau^+ L^{\frac{k-1}{2}} z, \Psi_\tau^+ L^{\frac{k-1}{2}} z\right\rangle_{L^2([0,\infty),Y)} \quad \forall\, z \in V_{\mathrm{Obs}},$$

and then $L^{n+1} < L^n$ since

$$\left\langle L^n z, z\right\rangle - \left\langle L^{n+1} z, z\right\rangle = 2\gamma \left\langle (I-\gamma E)\, \Psi_\tau^+ L^{\frac{n}{2}} z, \Psi_\tau^+ L^{\frac{n}{2}} z\right\rangle_{L^2([0,\infty),Y)} \quad \forall\, z \in V_{\mathrm{Obs}},$$

and the right-hand side of this equality is strictly positive from (3.11). In particular, this implies that the sequence $(\|L^n z\|)_{n\in\mathbb{N}}$ is strictly decreasing, for all $z \in V_{\mathrm{Obs}}$. Indeed, $\|L^n z\|^2 = \left\langle L^{2n} z, z\right\rangle > \left\langle L^{2(n+1)} z, z\right\rangle = \left\|L^{n+1} z\right\|^2$.
It remains to show that for all $z \in V_{\mathrm{Obs}}$, 0 is the limit of $(\|L^n z\|)_{n\in\mathbb{N}}$. As a decreasing sequence of positive operators on the Hilbert space V_{Obs}, Lemma 12.3.2 of [39] shows that the sequence converges in $\mathcal{L}(V_{\mathrm{Obs}})$ to a positive operator $L_\infty \in \mathcal{L}(V_{\mathrm{Obs}})$ such that

$$\lim_{n\to\infty} L^n z = L_\infty z \quad \forall\, z \in V_{\mathrm{Obs}},$$

and satisfying $L_\infty \leq L^n$ for all $n \in \mathbb{N}$. We have for all $z_1, z_2 \in V_{\mathrm{Obs}}$

$$\begin{aligned}\left\langle L_\infty^2 z_1, z_2\right\rangle &= \langle L_\infty z_1, L_\infty z_2\rangle \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \langle L^n z_1, L^m z_2\rangle \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \left\langle L^{n+m} z_1, z_2\right\rangle \\ &= \langle L_\infty z_1, z_2\rangle\,,\end{aligned}$$

which shows that $L_\infty^2 = L_\infty$. Furthermore, we have for all $z \in V_{\mathrm{Obs}}\setminus\{0\}$

$$\|L_\infty z\|^2 = \left\langle L_\infty^2 z, z\right\rangle = \langle L_\infty z, z\rangle \leq \left\langle L^2 z, z\right\rangle = \|Lz\|^2 < \|z\|^2,$$

The above inequality comes from the first point of Proposition 3.9.
Suppose now that Ran $L_\infty \neq \{0\}$. Then, there exists $z \in V_{\mathrm{Obs}}$ such that $L_\infty z \neq 0$ and then

$$\|L_\infty z\| = \|L_\infty^2 z\| < \|L_\infty z\|,$$

which is impossible. Thus Ran $L_\infty = \{0\}$, or in other words $L_\infty \equiv 0$. This shows that

$$\lim_{n\to\infty} L^n \Pi z = 0 \quad \forall\, z \in X.$$

We conclude using Corollary 3.7

$$\Pi\left(z_n^-(0)-z_0\right)=\Pi\left(\mathbb{T}_\tau^-\mathbb{T}_\tau^+\right)^n\left(z_0^+-z_0\right)=L^n\Pi\left(z_0^+-z_0\right)\underset{n\to\infty}{\longrightarrow}0\quad\forall\, z_0^+, z_0\in X.$$

Proof (Proof of Corollary 1.2) Using (3.1) and (3.2), we rewrite $z_1^-(0)$. We have for all $z_0, z_0^+ \in X$

$$z_1^+(\tau)=\mathbb{S}_\tau z_0^+-\gamma\Phi_\tau\Psi_\tau^+\left(z_0^+-z_0\right),$$

and

$$z_1^-(0)=\mathbb{S}_{-\tau}z_1^+(\tau)-\gamma\Phi_\tau^d\Psi_\tau^-\left(z_1^+(\tau)-\mathbb{S}_\tau z_0\right).$$

Substituting the first equality into the second one, we obtain

$$z_1^-(0)=z_0^+-\gamma\mathbb{S}_{-\tau}\Phi_\tau\Psi_\tau^+\left(z_0^+-z_0\right)-\gamma\Phi_\tau^d\Psi_\tau^-\left(z_1^+(\tau)-\mathbb{S}_\tau z_0\right).$$

From $\mathbb{S}_{-\tau}\Phi_\tau=\Phi_\tau^d\,\text{Я}_\tau$ and $\Phi_\tau^d=\Psi_\tau^*\,\text{Я}_\tau$, we get that, for all $z_0^+, \theta\in X$

$$\left\langle z_1^-(0),\theta\right\rangle=\left\langle z_0^+,\theta\right\rangle-\gamma\left\langle\Psi_\tau^+\left(z_0^+-z_0\right),\Psi_\tau\theta\right\rangle-\gamma\left\langle\text{Я}_\tau\Psi_\tau^-\left(z_1^+(\tau)-\mathbb{S}_\tau z_0\right),\Psi_\tau\theta\right\rangle.$$

This implies that

$$\left\langle z_1^-(0),\theta\right\rangle=0\quad\forall\, z_0^+\in V_{\text{Obs}},\theta\in V_{\text{Unobs}}.$$

In other words, for all $z_0^+\in V_{\text{Obs}}$, $z_1^-(0)\in V_{\text{Obs}}$. We can iterate the cycle of forward–backward observers and obtain that

$$z_0^+\in V_{\text{Obs}}\implies z_n^-(0)\in V_{\text{Obs}}\quad\forall\, n\in\mathbb{N}.$$

We apply Theorem 1.1 with $z_0^+\in V_{\text{Obs}}$ and the previous result to conclude. □

4 Example

In this section, we investigate a wave equation with colocated Dirichlet control and observation. This system is known to be well posed (see for instance Guo and Zhang [22]). Many other examples fitting into the framework of this paper can be found in the literature. We can mention another work of Guo and Zhang [23] for the wave equation with partial Dirichlet control and colocated observation with variable coefficients, the work of Chapelle et al. [9] on the wave equation with distributed observation, of Guo and Shao [21] for both non-uniform Schrödinger and Euler–Bernoulli equations with boundary control and observation, and of Curtain and Weiss [12,43] for the Rayleigh beam equation.

Let $\Omega \in \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and Γ_0 and Γ_1 are relatively open in $\partial\Omega$. Let Δ be the Dirichlet Laplacian, and ν the unit normal vector of Γ_1 pointing towards the exterior of Ω. We consider

$$\begin{cases} \ddot{w}(x,t) - \Delta w(x,t) = 0 & \forall\, x \in \Omega, \quad t > 0, \\ w(x,t) = 0 & \forall\, x \in \Gamma_0, \quad t > 0, \\ w(x,t) = u(x,t) & \forall\, x \in \Gamma_1, \quad t > 0, \\ w(x,0) = w_0(x) & \forall\, x \in \Omega, \\ \dot{w}(x,0) = w_1(x) & \forall\, x \in \Omega, \end{cases} \tag{4.1}$$

with u the input function (the control), and (w_0, w_1) the initial state. We observe this system on Γ_1, leading to the measurement

$$y(x,t) = -\frac{\partial(-\Delta)^{-1}\dot{w}(x,t)}{\partial\nu} \quad \forall\, x \in \Gamma_1, t > 0. \tag{4.2}$$

Guo and Zhang [22, Theorem 1.1] proved that this evolution partial differential equation can be represented by a well posed linear system Σ with state space $X = L^2(\Omega) \times H^{-1}(\Omega)$ and $U = Y = L^2(\Gamma_1)$, and that the operators (A, B, C) satisfy $A^* = -A$ and $B = C^*$. More precisely, there exist A_0 (namely $-\Delta$) a positive definite self-adjoint operator such that

$$A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix},$$

and $C_0 \in \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{1}{2}}\right), Y\right)$ such that

$$C = [0 \quad C_0].$$

Moreover, the transfer function of this system is given by

$$\mathbf{G}(s) = sC_0\left(s^2 I + A_0\right)^{-1} C_0^* \quad \forall\, s \in \mathbb{C}_0.$$

Thus, since $0 \in \rho(A)$, Remark 2.18 gives

$$E = -\frac{1}{2}\left(\mathbf{G}^*(0) + \mathbf{G}(0)\right) = 0.$$

In particular, the value of κ in Theorem 1.1 is equal to infinity.

Theorem 4.1 *Let $\gamma > 0$ and $\tau > 0$, $(w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ be the initial state of (4.1), $u \in L^2([0,\tau), L^2(\Gamma_1))$ its input function, w its solution, and y its output, given by (4.2). Denote, for all $n \geq 1$, w_n^+ and w_n^- the respective solutions of*

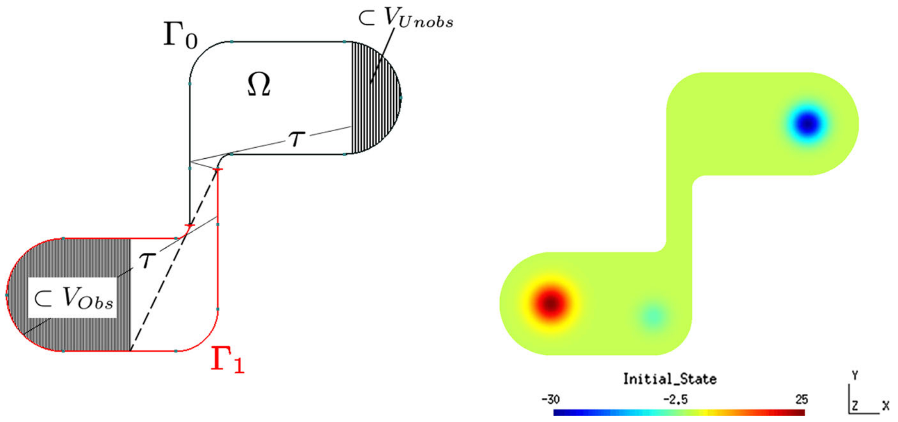

Fig. 1 An example of configuration in two dimensions and the initial state to reconstruct

$$
\begin{cases}
\ddot{w}_n^+(x,t) - \Delta w_n^+(x,t) = 0 & \forall\, x \in \Omega, \quad t \in (0,\tau),\\
w_n^+(x,t) = 0 & \forall\, x \in \Gamma_0, \quad t \in (0,\tau),\\
w_n^+(x,t) = \gamma \dfrac{\partial(-\Delta)^{-1}\dot{w}_n^+(x,t)}{\partial\nu} + \gamma y(x,t) + u(x,t) & \forall\, x \in \Gamma_1, \quad t \in (0,\tau),\\
w_1^+(x,0) = 0 \quad \dot{w}_1^+(x,0) = 0 & \forall\, x \in \Omega,\\
w_n^+(x,0) = w_{n-1}^-(x,0), \quad \dot{w}_n^+(x,0) = \dot{w}_{n-1}^-(x,0) & \forall\, x \in \Omega, \quad n \geq 2,
\end{cases}
$$

and

$$
\begin{cases}
\ddot{w}_n^-(x,t) - \Delta w_n^-(x,t) = 0 & \forall\, x \in \Omega, \quad t \in (0,\tau),\\
w_n^-(x,t) = 0 & \forall\, x \in \Gamma_0, \quad t \in (0,\tau),\\
w_n^-(x,t) = \gamma \dfrac{\partial(-\Delta)^{-1}\dot{w}_n^-(x,t)}{\partial\nu} - \gamma y(x,t) + u(x,t) & \forall\, x \in \Gamma_1, \quad t \in (0,\tau),\\
w_n^-(x,\tau) = w_n^+(x,\tau), \quad \dot{w}_n^-(x,\tau) = \dot{w}_n^+(x,\tau) & \forall\, x \in \Omega.
\end{cases}
$$

Denote Π the orthogonal projector from $L^2(\Omega) \times H^{-1}(\Omega)$ onto $V_{\mathrm{Obs}} = \overline{\mathrm{Ran}\,\Phi_\tau^d}$ (we do not show it explicitly). Then, from Corollary 1.2, we have

$$
\left\| \begin{pmatrix} w_n^-(x,0) \\ \dot{w}_n^-(x,0) \end{pmatrix} - \Pi \begin{pmatrix} w_0(x) \\ w_1(x) \end{pmatrix} \right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \underset{n\to\infty}{\longrightarrow} 0.
$$

Furthermore, the decay is exponential if and only if $\mathrm{Ran}\,\Phi_\tau^d$ *is closed in $L^2(\Omega) \times H^{-1}(\Omega)$.*

To illustrate Theorem 4.1, consider the configuration on the left of Fig. 1 and let us try to reconstruct the initial state on the right of Fig. 1, constituted of three bumps, with null initial velocity. For simplicity sake, we take $u \equiv 0$.

Then, we choose $\tau > 0$ such that, using the geometric optic rays (see Bardos et al. [5]), we can reconstruct all initial data with support included in the left striped part, and that no information can be obtained from the initial data with support included in the right striped part. In particular, we cannot expect to reconstruct the bump in the right top part of Ω.

Remark 4.2 It is well-known that uniform controllability/observability (with respect to the mesh size parameters) may fail after discretization (see for instance Zhang et al. [46]) due to high-frequency spurious modes. Using a numerical viscosity method, Ervedoza and Zuazua [14] proposed a time discretization preserving the uniform (in the time parameter) exponential stability of a damped wave equation.

More recently, García and Takahashi [17] used a finite-difference discretization in space for a one-dimensional wave equation. To avoid restrictions on the number of steps with respect to the mesh size, they add a vanishing viscosity in the numerical observers. They prove an estimate of the errors with respect to the mesh size and to the number of steps in the algorithm of [32].

In the case studied here, where we do not have exact observability, it is not clear if such a process can be used to tackle the spurious modes. Indeed, a further investigation of the discretization of V_{Obs} and its stability under the discretized algorithm should be done.

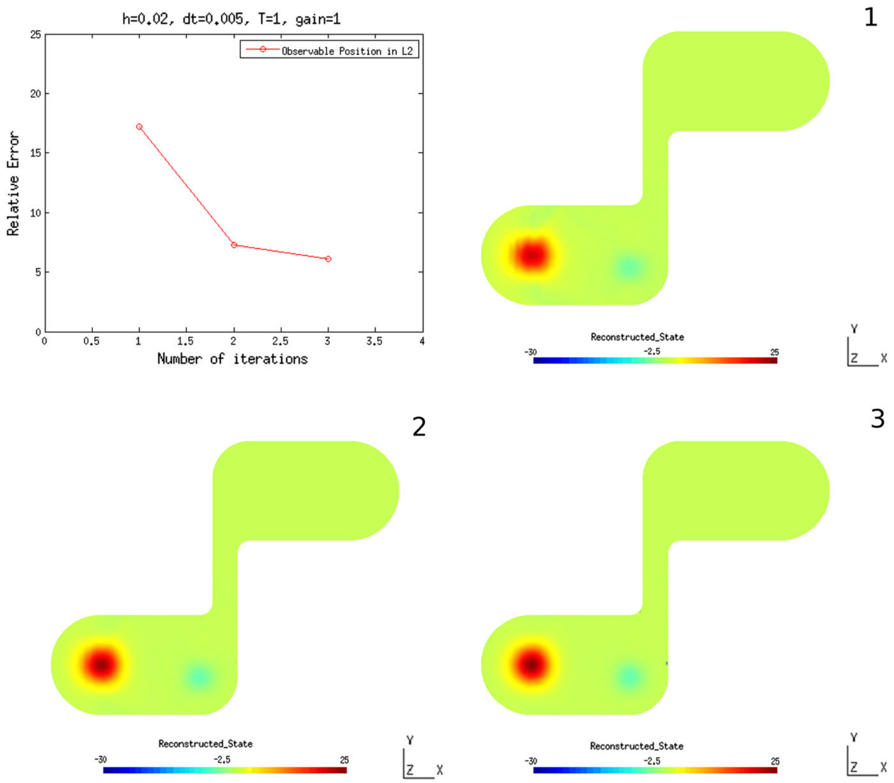

Fig. 2 Relative error of the "observable part of the position" in $L^2(\Omega)$ and the reconstructions obtained after the first, second and third iterations

Remark 4.3 In presence of noisy measurement, we do not know if the stability of V_{Obs} under the discretized algorithm is preserved. It is more likely that this stability fails, leading to a deterioration of the reconstruction.

Using GMSH [20] and GetDP [13], we have implemented the algorithm with finite elements in space (parameter $h = 0.02$) and an unconditionally stable Newmark scheme in time (parameter $\Delta t = 0.005$), with $\gamma = 1$ and a time of observation $\tau = 1$.

We have obtained Fig. 2, where we can see the efficiency of the algorithm to reconstruct "the observable part" of the initial data (we take here the truncation on the bottom part of the figure). After only three iterations, we reach 6 % of relative error in $L^2(\Omega)$.

References

1. Auroux D, Blum J (2005) Back and forth nudging algorithm for data assimilation problems. C R Math Acad Sci Paris 340(12):873–878
2. Auroux D, Blum J (2008) A nudging-based data assimilation method for oceanographic problems: the back and forth nudging (bfn) algorithm. Nonlinear Proc Geophys 15:305–319
3. Auroux D, Blum J, Nodet M (2011) Diffusive back and forth nudging algorithm for data assimilation. C R Math Acad Sci Paris 349(15–16):849–854
4. Baras JS, Bensoussan A (1987) On observer problems for systems governed by partial differential equations. No. SCR TR 86–47. Maryland University College Park, pp 1–26
5. Bardos C, Lebeau G, Rauch J (1992) Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J Control Optim 30(5):1024–1065
6. Bensoussan A (1993) Filtrage optimal des Systèmes Linéaires. Dunod
7. Blum J, Le Dimet F-X, Navon IM (2009) Data assimilation for geophysical fluids. In: Handbook of numerical analysis. Vol. XIV. Special volume: computational methods for the atmosphere and the Oceans, vol 14 of Handb. Numer. Anal. Elsevier, North-Holland, pp 385–441
8. Brezis H (2011) Functional analysis. Sobolev spaces and partial differential equations. Universitext. Springer, New York
9. Chapelle D, Cîndea N, De Buhan M, Moireau P (2012) Exponential convergence of an observer based on partial field measurements for the wave equation. Math Probl Eng. Art. ID 581053, 12
10. Couchouron J-F, Ligarius P (2003) Nonlinear observers in reflexive Banach spaces. ESAIM Control Optim Calc Var 9:67–104 (electronic)
11. Curtain RF, Weiss G (1989) Well, posedness of triples of operators (in the sense of linear systems theory). In: Control and estimation of distributed parameter systems (Vorau, (1988) volume 91 of Internat. Ser. Numer. Math. Birkhäuser, Basel, pp 41–59
12. Curtain RF, Weiss G (2006) Exponential stabilization of well-posed systems by colocated feedback. SIAM J Control Optim 45(1):273–297 (electronic)
13. Dular P, Geuzaine C GetDP reference manual: the documentation for GetDP, a general environment for the treatment of discrete problems. http://www.geuz.org/getdp/
14. Ervedoza S, Zuazua E (2009) Uniformly exponentially stable approximations for a class of damped systems. J Math Pures Appl (9) 91(1):20–48
15. Fink M (1992) Time reversal of ultrasonic fields-basic principles. IEEE Trans Ultrason Ferro Electric Freq Control 39:555–556
16. Fink M, Cassereau D, Derode A, Prada C, Roux O, Tanter M, Thomas J-L, Wu F (2000) Time-reversed acoustics. Rep Prog Phys 63(12):1933–1995
17. García GC, Takahashi T (2013) Numerical observers with vanishing viscosity for the 1d wave equation. Adv Comput Math 1–35
18. Gebauer B, Scherzer O (2008) Impedance-acoustic tomography. SIAM J Appl Math 69(2):565–576
19. Gejadze IY, Le Dimet F-X, Shutyaev V (2010) On optimal solution error covariances in variational data assimilation problems. J Comput Phys 229(6):2159–2178
20. Geuzaine C, Remacle J-F (2009) Gmsh: a 3-D finite element mesh generator with built-in pre- and post-processing facilities. Int J Numer Methods Eng 79(11):1309–1331

21. Guo B-Z, Shao Z-C (2012) Well-posedness and regularity for non-uniform Schrödinger and Euler-Bernoulli equations with boundary control and observation. Quart Appl Math 70(1):111–132
22. Guo B-Z, Zhang X (2005) The regularity of the wave equation with partial Dirichlet control and colocated observation. SIAM J Control Optim 44(5):1598–1613
23. Guo B-Z, Zhang Z-X (2007) On the well-posedness and regularity of the wave equation with variable coefficients. ESAIM Control Optim Calc Var 13(4):776–792
24. Ito K, Ramdani K, Tucsnak M (2011) A time reversal based algorithm for solving initial data inverse problems. Discret Contin Dyn Syst Ser S 4(3):641–652
25. Krstic M, Guo B-Z, Smyshlyaev A (2011) Boundary controllers and observers for the linearized Schrödinger equation. SIAM J Control Optim 49(4):1479–1497
26. Kuchment P, Kunyansky L (2008) Mathematics of thermoacoustic tomography. Eur J Appl Math 19(2):191–224
27. Le Dimet F-X, Shutyaev V, Gejadze IY (2006) On optimal solution error in variational data assimilation: theoretical aspects. Russ J Numer Anal Math Model 21(2):139–152
28. Liu K (1997) Locally distributed control and damping for the conservative systems. SIAM J Control Optim 35(5):1574–1590
29. Luenberger D (1964) Observing the state of a linear system. IEEE Trans Mil Electron 8:74–80
30. Moireau P, Chapelle D, Le Tallec P (2008) Joint state and parameter estimation for distributed mechanical systems. Comput Methods Appl Mech Eng 197(6–8):659–677
31. Phung KD, Zhang X (2008) Time reversal focusing of the initial state for Kirchhoff plate. SIAM J Appl Math 68(6):1535–1556
32. Ramdani K, Tucsnak M, Weiss G (2010) Recovering the initial state of an infinite-dimensional system using observers. Automatica 46:1616–1625
33. Salamon D (1989) Realization theory in Hilbert space. Math Syst Theory 21:147–164
34. Shutyaev V-P, Gejadze IY (2011) Adjoint to the Hessian derivative and error covariances in variational data assimilation. Russ J Numer Anal Math Model 26(2):179–188
35. Smyshlyaev A, Krstic M (2005) Backstepping observers for a class of parabolic PDEs. Syst Control Lett. 54(7):613–625
36. Staffans O, Weiss G (2002) Transfer functions of regular linear systems. II. The system operator and the Lax-Phillips semigroup. Trans Am Math Soc 354(8):3229–3262
37. Staffans O, Weiss G (2004) Transfer functions of regular linear systems. III. Inversions and duality. Integral Equ Oper Theory 49(4):517–558
38. Teng JJ, Zhang G, Huang SX (2007) Some theoretical problems on variational data assimilation. Appl Math Mech 28(5):581–591
39. Tucsnak M, Weiss G (2009) Observation and control for operator semigroups. Birkhäuser Verlag, Basel
40. Weiss G (1989) The, representation of regular linear systems on Hilbert spaces. In: Control and estimation of distributed parameter systems (Vorau, (1988) vol 91 of Internat. Ser. Numer. Math. Birkhäuser, Basel, pp 401–416
41. Weiss G (1994) Regular linear systems with feedback. Math Control Signals Syst 7(1):23–57
42. Weiss G (1994) Transfer functions of regular linear systems. I. Characterizations of regularity. Trans Am Math Soc 342(2):827–854
43. Weiss G, Curtain RF (2008) Exponential stabilization of a Rayleigh beam using collocated control. IEEE Trans Autom Control 53(3):643–654
44. Weiss G, Rebarber R (2000) Optimizability and estimatability for infinite-dimensional linear systems. SIAM J Control Optim. 39(4):1204–1232 (electronic)
45. Weiss G, Staffans OJ, Tucsnak M (2001) Well-posed linear systems–a survey with emphasis on conservative systems. Int J Appl Math Comput Sci 11(1):7–33
46. Zhang X, Zheng C, Zuazua E (2009) Time discrete wave equations: boundary observability and control. Discret Contin Dyn Syst 23(1–2):571–604